# On the consistency relation of the 3-point function in single field inflation

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**Abstract**

The consistency relation for the 3-point function of the CMB is a very powerful observational signature which is believed to be true for every inflationary model in which there is only one dynamical degree of freedom. Its importance relies on the fact that deviations from it might be detected in next generation experiments, allowing us to rule out all single field inflationary models. After making more precise the already existing proof of the consistency relation, we use a recently developed effective field theory for inflationary perturbations to provide an alternative and very explicit proof valid at leading non trivial order in slow roll parameters.

# 1 Introduction

In the last few years there has been great progress in understanding the non-Gaussianity of the primordial spectrum of density fluctuations. Starting from Maldacena's first full computation of the non-Gaussian features in single field slow roll inflation [1], several alternative models have been proposed that produce a large and in principle detectable level of non-Gaussianities [2, 3, 4, 6] through different mechanisms for generating density fluctuations in the quasi de Sitter inflationary phase. At the same time, from the experimental side, the WMAP satellite has allowed for a huge improvement in our measurement of the properties of the CMB. Observations seem to confirm the generic predictions of standard slow roll inflation [10]. Limits on the primordial non-Gaussianity of the CMB have been significantly improved [11], but for the moment the data are consistent with a non-Gaussian signal.

The fact that the CMB seems to be rather Gaussian means that the non-Gaussian component must be rather small. This makes it clear that the most important observable for non-Gaussianities will be the 3-point function of density perturbations [12]

$$\langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \zeta_{\vec{k}_3} \rangle \tag{1}$$

where $\zeta_{\vec{k}_1}$ is the density fluctuation of comoving slices in Fourier space.

As pointed out in [14], due to symmetry reasons, the 3-point function is a real function of *two* variables. While on one hand this means that it contains a lot of information about the inflationary model, on the other hand this also means that there really could be a large number of different

shapes of the 3-point function. For this reason, model independent characterization of the 3-point function are very useful.

To this end, it was pointed out in [1, 15] that in all cases in which there is only one dynamical field that is important during inflation, the three point function is connected to the two point function and its deviation from scale invariance in a particular geometrical limit . In other words, we have:

$$\lim_{k_1\to 0}\langle\zeta_{\vec{k}_1}\zeta_{\vec{k}_2}\zeta_{\vec{k}_3}\rangle = -(2\pi)^3\delta^3(\sum_i \vec{k}_i)P_{k_1}P_{k_3}\frac{d\log k_3^3 P_{k_3}}{d\log k_3}\ , \tag{2}$$

where

$$\langle\zeta_{\vec{k}_i}\zeta_{\vec{k}_j}\rangle = (2\pi)^3\delta^3(\vec{k}_i+\vec{k}_j)P_{k_i}\ . \tag{3}$$

Let us comment briefly on what this relationship means. Because of translation invariance, the spatial momenta $\vec{k}_1, \vec{k}_2, \vec{k}_3$ must form a closed triangle. The consistency relation says that in the limit in which one of the sides of this triangle (say $\vec{k}_1$) goes to zero – and therefore the other two sides become equal and opposite ($\vec{k}_1 \simeq -\vec{k}_2$), and the triangle becomes squeezed – the three point function becomes proportional to the two point function of the long wavelength modes times the two point function of the short wavelength mode times its deviation from scale invariance. There are no free parameters: in this limit the three point function can be fully specified in terms of the two point function. Notice also that, as we will explicitly show later, the consistency relation holds without any slow roll approximation. We emphasize that our only assumption is that there is only one relevant single clock field during the cosmological history.

Experimental limits on non-Guassianities are generically given in terms of a scalar variable $f_{\rm NL}$ [13, 14] which gives an amplitude of the 3-point function in the squeezed limit of the form:

$$\lim_{k_1\to 0}\langle\zeta_{\vec{k}_1}\zeta_{\vec{k}_2}\zeta_{\vec{k}_3}\rangle = (2\pi)^3\delta^3(\sum_i \vec{k}_i)\ 4 f_{\rm NL}P_{k_1}P_{k_3} \tag{4}$$

In the limit of small deviation from de Sitter, the consistency relation predicts that the level of $f_{\rm NL}$ in the squeezed configuration should be of order of the slow roll parameters $\mathcal{O}(1/100)$. Current limits from WMAP 3-year data give an $f_{NL} \lesssim 100$ [11], and the Planck satellite is expected to constrain $f_{\rm NL}$ at the level of a few. We therefore realize that it is probably impossible, at least in the foreseeable future, to experimentally verify the consistency relation.

This does not mean that the consistency relation is useless. It is instead a very powerful instrument in the opposite regime in which it is proven to be experimentally not satisfied. *No* model of inflation with only one dynamical degree of freedom (which therefore acts as a physical 'clock' for exiting the quasi de Sitter phase) can predict a non-Gaussian signal in the squeezed limit of detectable level, meaning that a detection of a signal in this limit would allow us to *rule out* all single field inflationary models.

The assumption of a single 'clock' is really essential, as we will explain, and in fact there are inflationary models with more than one field that violate the consistency relation, and predict a detectable level on non-Gaussianity in the squeezed limit (see for example [5, 6]). Outside of the inflationay paradigm, the recently proposed new bouncing cosmology [7, 8, 9], though less compelling than inflation, predicts a potentially detectable non-Gaussian signal that violates the consistency relation [9]. All of this is somewhat reassuring from the theoretical point of view, as it tells us that we do have some hope of detecting a deviation from the consistency relation in the near future.

The available arguments that prove the consistency relation [1, 15] are rigorous, but suffer from not being very explicit, casting some doubt on their validity whenever some explicit calculation seems to show that it is violated. The purpose of the present paper is to explicitly verify the consistency relation for all single clock models at leading non trivial order in slow roll parameters. We can do this because we can exploit a recently developed effective theory [16, 17] which describes fluctuations around a FRW cosmology for every single field model. The assumptions behind the validity of this effective theory are exactly the same for which the consistency relation applies, and this allows us to verify the consistency relation in full generality. Even if general proofs do exist [1, 15], we consider such an explicit verification worthwhile for the outlined importance of the consistency relation as being able to rule out all single field models of inflation.

The paper is structured as follows. In sec. 2 we review and further formalize the generic proof of the consistency relation. In particular we highlight the fact that it is valid beyond any slow roll approximation. In sec. 3 we begin the verification of the consistency relation at leading order in slow roll parameters after having briefly introduced the effective theory we will use. We first study a very large class of models which includes for example all the models with a Lagrangian for a single scalar field of the form $P(-(\partial\phi)^2, \phi)$, explicitly verifying the consistency relation at first order in slow roll parameters. Then we go on to study some more 'exotic' inflationary models that appear in our effective theory within some simplified assumption. In sec. 4 we summarize and conclude.

# 2 Formal proof of the consistency relation [1]

For simplicity we concentrate on only the scalar fluctuations and neglect tensor modes [2]. Using the $\zeta$ variable to describe scalar fluctuations, the metric for fluctuations around a FRW universe takes the form:

$$ds^2 = -N^2 dt^2 + \hat{g}_{ij}(dx^i + N^i dt)(dx^j + N^j dt) \ , \tag{5}$$

with

$$\hat{g}_{ij} = a(t)^2 e^{2\zeta} \delta_{ij} \ , \tag{6}$$

where we have used the ADM parameterization. In this gauge the matter is taken to be unperturbed, fixing the time diffeomorphisms in this way.

Let us introduce two facts that will be proven to be true later. First, when a mode goes outside the horizon ($\omega \ll H$), if there is only one degree of freedom, the ADM variable $N$ and $N^i$ defined in the gauge of (5), go to their unperturbed values (respectively 1 and 0) . In this limit the metric becomes

$$ds^2 = -dt^2 + a(t)^2 e^{2\zeta(x)} dx_i dx_i \tag{7}$$

with $\zeta$ constant in time. Since in this limit we can neglect the gradient terms, we can re-absorb the $\zeta$ fluctuation in a 'local' rescaling of the coordinates $x' = e^{\zeta(x)} x$. In regions of space separated by large distances, the metric becomes the one of unperturbed FRW universes, each one characterized

[1] The results of this section are obtained in collaboration with Paolo Creminelli.

[2] We make this assumption for simplicity's sake and also because the 3-point function of scalar fluctuations is at least for the moment the most important from the observational point of view. This assumption is not necessary for the proof of the consistency relation, as it was previously noted in [1, 18], and more general fluctuations can be easily included.

by a different rescaling of the coordinate (or equivalently of the scale factor). This is sometimes called the parallel universe description of the inflationary perturbation outside of the horizon. There is a quite intuitive reason why on large scales the metric takes the form (7). Once we can neglect gradients, each region of the universe evolves exactly in the same way, as the inflationary solution is an attractor. As a consequence, the metric on large scales has to approach the one of the unperturbed FRW universe. The only difference among the well separated observers is how much each region has expanded with respect to the other, and it is this difference that remains constant.

The second fact that is important for us is that once a mode goes outside the horizon, it becomes classical, in the sense that $[\zeta_{\vec{k}}, \dot{\zeta}_{\vec{k}'}] \to 0$ exponentially fast. So for measurements which only involve $\zeta$ or $\dot{\zeta}$ we can treat the mode as a classical variable. Proof of this with different approaches can be found for example in [19, 20].

Let us now concentrate on the three point function of eq. (2). We can imagine going to the limit where all the three modes are well outside the horizon, so that we can treat them as classical. We are interested in the regime where $k_1 \ll k_2, k_3$. $\zeta_{\vec{k}_1}$ is therefore a background mode for $\zeta_{\vec{k}_2}$ and $\zeta_{\vec{k}_3}$. One can therefore compute the three point function in a two step process: first compute the two point function of $\zeta_{\vec{k}_2}$ and $\zeta_{\vec{k}_3}$ in a background $\zeta^B$:

$$\langle \zeta_{\vec{k}_2} \zeta_{\vec{k}_3} \rangle_{\zeta^B} \ , \tag{8}$$

and then correlate this result with the value of the background field $\zeta_{\vec{k}_1}$. It is useful to compute this two point function in real space and so compute $\langle \zeta_{\vec{x}_2} \zeta_{\vec{x}_3} \rangle$ on the background $\zeta^B(\vec{x})$. The scale of variation of the background is much larger than $|\vec{x}_2 - \vec{x}_3|$. From this point of view what we are computing is the short scale average on a given realization of the background. Then, we will correlate with the background and average over it. Expanding the short scale two point function in powers of the background, we obtain:

$$\langle \zeta\zeta \rangle_B(\vec{x}_2, \vec{x}_3) \simeq \langle \zeta\zeta \rangle_0(|\vec{x}_2 - \vec{x}_3|) + \zeta^B(\frac{\vec{x}_2 + \vec{x}_3}{2}) \left( \frac{d}{d\zeta^B} \langle \zeta\zeta \rangle_B(|\vec{x}_2 - \vec{x}_3|) \right)\Big|_0 \ , \tag{9}$$

where the subscript 0 means that the quantity is evaluated on the vacuum, i.e. without the background wave. On the background the two point function does not depend only on the distance between the two points, but also on their position on the background. Since the points are very close with respect to the the typical variation length of the background we can evaluate the background at the middle point $(\vec{x}_2 + \vec{x}_3)/2$. Corrections to this are sub-leading in the squeezed limit expansion. Notice that no slow roll approximation has been done: we have just expanded in powers of the small background field. The background modulates the amplitude of the two point function; as $\zeta$ is equivalent to a rescaling of the spatial coordinates, we can trade the derivative with respect to $\zeta^B$ for a derivative with respect to the log-distance between the points:

$$\langle \zeta\zeta \rangle_B(\vec{x}_2, \vec{x}_3) \simeq \langle \zeta\zeta \rangle_0(|\vec{x}_2 - \vec{x}_3|) + \int \frac{d^3k}{(2\pi)^3} \, \zeta^B(\vec{k}) e^{i\vec{k}\cdot(\vec{x}_2 + \vec{x}_3)/2} \frac{d}{d\log(|\vec{x}_2 - \vec{x}_3|)} \langle \zeta\zeta \rangle_0(|\vec{x}_2 - \vec{x}_3|) \ , \tag{10}$$

where we have written $\zeta^B$ in Fourier space:

$$\zeta^B(\vec{x}) = \int \frac{d^3k}{(2\pi)^3} \zeta^B(\vec{k}) e^{i\vec{k}\cdot\vec{x}} \ , \tag{11}$$

because this will soon be useful [3]. Now, we can do the Fourier transform with respect to $\vec{x}_2$ and $\vec{x}_3$. The result can be expressed in terms of $\vec{k}_L = \vec{k}_2 + \vec{k}_3$ and $\vec{k}_S = (\vec{k}_2 - \vec{k}_3)/2$. Here $_L$ and $_S$ stand for long and short wavelength. The derivative with respect the log-distance can be integrated by parts to obtain a derivative with respect to the $\log k_S$. After this algebra we obtain:

$$\langle\zeta\zeta\rangle_B(\vec{k}_2, \vec{k}_3) = \langle\zeta\zeta\rangle_0(k_S) - \zeta^B(\vec{k}_L)\frac{1}{k_S^3}\frac{d}{d\log k_S}[k_S^3\langle\zeta\zeta\rangle_0(k_S)] \ , \tag{12}$$

where we have used that $\vec{k}_2 \simeq -\vec{k}_3 \simeq \vec{k}_S$ in the squeezed limit up to sub-leading corrections. We can now multiply by $\zeta^{(B)}_{\vec{k}_1}$ and take the average. The piece which is independent of the background gives no contribution, and we are left with:

$$\begin{aligned}
\langle\zeta^{(B)}(\vec{k}_1)\langle\zeta\zeta\rangle_B(\vec{k}_2, \vec{k}_3)\rangle &= -\langle\zeta^{(B)}(\vec{k}_1)\zeta^B(\vec{k}_L)\rangle\frac{1}{k_S^3}\frac{d}{d\log k_S}[k_S^3\langle\zeta\zeta\rangle_0(k_S)] && (13)\\
&= -(2\pi)^3\delta^3(\vec{k}_1 + \vec{k}_L)P_{k_1}P_{k_2}\frac{d\log[k_2^3 P_{k_2}]}{d\log k_2} && (14)\\
&= -(2\pi)^3\delta^3(\vec{k}_1 + \vec{k}_2 + \vec{k}_3)P_{k_1}P_{k_2}\frac{d\log[k_2^3 P_{k_2}]}{d\log k_2} && (15)
\end{aligned}$$

where we have used eq.(3).

We have thus obtained eq.(2) as we wished. We stress again that *no* slow roll expansion was done in this proof (which remains valid for example even if there are sharp features in the potential), and that the only important assumption was that there is only one clock field. This has allowed us to expand the short scale 2-point function in powers only of the background field $\zeta^B$ in eq.(9) and not also in terms of some other field.

We now are going to briefly introduce our effective field theory for one clock inflation, and then to explicitly verify that the consistency relation holds with a direct calculation.

# 3 Explicit Verification with the effective theory for inflation

## 3.1 The effective Lagrangian for single field inflation

In this section we briefly introduce the effective action for single clock inflation that we will use to verify the consistency relation at leading non trivial order in slow roll parameters. This effective action was developed in [16, 17] and we refer the reader to those papers for a detailed explanation. The construction of the effective theory is based on the following consideration. In a quasi de Sitter background with only one relevant degree of freedom, there is a privileged spatial slicing, given by the physical clock which allows us to smoothly connect to a decelerated hot Big Bang evolution. The slicing is usually realized by a time evolving scalar $\phi(t)$. To describe perturbations around this solution one can choose a gauge where the privileged slicing coincides with surfaces of constant $t$, *i.e.* $\delta\phi(\vec{x}, t) = 0$. In this gauge there are no explicit scalar perturbations, but only metric fluctuations. As time diffeomorphisms have been fixed and are not a gauge symmetry anymore,

[3] Notice that this is our convention for the Fourier transform.

the graviton now describes three degrees of freedom: the scalar perturbation has been eaten by the metric. One therefore can build the most generic effective action with operators that are functions of the metric fluctuations and that are invariant under the linearly realized time dependent spatial diffeomorphisms. As usual with effective field theories, this can be done in a low energy expansion in fluctuations of the fields and derivatives. We obtain for the matter Lagrangian [16, 17]:

$$\begin{aligned} S_{\rm matter} = \int d^4x\, \sqrt{-g} \quad & \Big[ -M_{\rm Pl}^2 \dot H \frac{1}{N^2} - M_{\rm Pl}^2 \left(3H^2 + \dot H\right) + \\ & \frac{M(t)^4}{2!}\left(\frac{1}{N^2}-1\right)^2 + \frac{c_3(t) M(t)^4}{3!}\left(\frac{1}{N^2}-1\right)^3 + \\ & + \frac{d_1(t)}{2} M^3(t)\delta N \delta E^i{}_i - \frac{d_2(t)}{2} M(t)^2 \delta E^i{}_i{}^2 - \frac{d_3(t)}{2} M(t)^2 \delta E^i{}_j \delta E^j{}_i + \ldots \Big] \ , \end{aligned} \tag{16}$$

where we have used the ADM formalism. We have defined:

$$\delta N = N - 1 \ , \tag{17}$$

and $\delta E_{ij} = E_{ij} - a^2 H \hat g_{ij}$ is the fluctuation in the quantity $E_{ij}$ which is related to the extrinsic curvature of hypersurfaces of constant $t$:

$$E_{ij} \equiv N K_{ij} = \frac{1}{2}[\partial_t \hat g_{ij} - \hat\nabla_i N_j - \hat\nabla_j N_i] \ . \tag{18}$$

Here $\hat\nabla_i$ is the derivative with respect to the spatial metric $\hat g_{ij}$.

At this point it is useful to reintroduce the full diff. invariance of the theory by reintroducing the Goldstone boson $\pi$ of time translation with the so called Stückelberg trick. This amounts to performing a time diffeomorphism in the Lagrangian of (16) of parameter $-\pi$, and then promoting $\pi$ to a field which shifts under time-diffeomorphisms:

$$\pi \to \tilde\pi(\tilde x(x)) = \pi(x) - \xi^0(\vec x, t) \ . \tag{19}$$

This procedure is explained more in detail in [17], and we refer to it for further details. Neglecting for the moment the terms that involve the extrinsic curvature, we obtain:

$$\begin{aligned} S_{\rm matter} = \int d^4x\, \sqrt{-g} \quad & \Big[ -M_{\rm Pl}^2 \dot H(t+\pi)\left(\frac{1}{N^2}\left(1+\dot\pi - N^i\partial_i\pi\right)^2 - \hat g^{ij}\partial_i\pi\partial_j\pi\right) \\ & -M_{\rm Pl}^2\left(3H^2(t+\pi) + \dot H(t+\pi)\right) + \\ & \frac{M(t+\pi)^4}{2}\left(\frac{1}{N^2}\left(1+\dot\pi - N^i\partial_i\pi\right)^2 - \hat g^{ij}\partial_i\pi\partial_j\pi - 1\right)^2 + \\ & \frac{c_3(t+\pi)\, M(t+\pi)^4}{6}\left(\frac{1}{N^2}\left(1+\dot\pi - N^i\partial_i\pi\right)^2 - \hat g^{ij}\partial_i\pi\partial_j\pi - 1\right)^3 + \ldots \Big] \ , \end{aligned} \tag{20}$$

Obviously, there is also the Einstein Hilbert action:

$$S_{\rm EH} = \frac{1}{2} M_{\rm Pl}^2 \int d^4x\, \sqrt{-g}\, R = \frac{1}{2} M_{\rm Pl}^2 \int d^3x\, dt\, \sqrt{\hat g}\, [N R^{(3)} + \frac{1}{N}(E^{ij}E_{ij} - E^i{}_i{}^2)] \ . \tag{21}$$

No $\pi$ appears explicitly in the Einstein Hilbert action after we perform the Stückelberg trick because the Einstein Hilbert action is already time diff. invariant.

The validity of this effective action is very general. It assumes only the presence of one degree of freedom which spontaneously breaks time translation and acts as a physical clock for the system. For example it reproduces all known models of single field inflation. We refer to [17] for a more general discussion of this point.

We are going to verify the consistency relation with our effective Lagrangian. Clearly, for such a complex Lagrangian, doing it in full generality is a difficult task because of the amount of algebra that this requires. In order to keep the complexity of the algebra at a minimum, in the next subsections we will verify it for some particular (and still very general) cases where each time we neglect the contribution of some specific operators.

## 3.2 Verification for the Lagrangian of the form $P(-(\partial\phi)^2, \phi)$

The Lagrangian (20) is already very general even though it omits all the operators which in unitary gauge involve the extrinsic curvature. It in fact reproduces the Lagrangian for all the models of inflation with a single scalar field and a Lagrangian of the form

$$S = \int d^4x \sqrt{-g}\; P(-(\partial\phi)^2, \phi) \ , \tag{22}$$

which are referred to as k-inflation [21]. This is easy to see if we write the Lagrangian in unitary gauge (which here means $\phi(\vec{x}, t) = \phi_0(t)$):

$$S = \int d^4x \sqrt{-g}\; P(\frac{\dot{\phi}_0^2}{N^2}, \phi_0(t)) \ , \tag{23}$$

which is of the form of (20). In App. A we explicitly perform the matching between the parameters in the Lagrangian (20) and the ones in (23).

### 3.2.1 The Lagrangian at first order in slow roll parameters

With only one scalar degree of freedom, it is necessary to integrate out the ADM variables $N$ and $N^i$ in order to find its effective action. Since these variables do not have kinetic terms, their equations of motion are algebraic. This is guaranteed to occur because of diff. invariance. Solving for $N$ and $N^i$ in terms of $\pi$, we can substitute these back into the Lagrangian. This process has been discussed extensively in [1, 4], and corresponds to removing gauge degrees of freedom. Indeed, counting all gauge degrees of freedom, there are exactly four scalars in the metric and one from the matter sector (namely, $\pi$). We shall fix time and space diffeomorphisms by choosing the gauge [4]:

$$\hat{g}_{ij} = a^2(t)\delta_{ij} \ . \tag{24}$$

which removes two scalar degrees of freedom from the metric. Then, we will solve for $N$ and $N^i$ removing two more, yielding one scalar degree of freedom $\pi$ in the final action.

[4]From here on we concentrate only on scalar fluctuations. This is enough for verifying the consistency condition of scalar perturbations at the leading order in slow roll parameters. For higher order calculations, one should include also tensor modes.

The equations of motion for $N$ and $N^i$ are

$$\frac{\partial(\mathcal{L}_{\rm EH}+\mathcal{L}_{\rm matt})}{\partial N^i} = 0 \, , \tag{25}$$

$$\frac{\partial(\mathcal{L}_{\rm EH}+\mathcal{L}_{\rm matt})}{\partial N} = 0 \, . \tag{26}$$

For the general properties of the ADM parameterization, these two equations allow us to express $N$ and $N^i$ in terms of $\pi$. As outlined in [1, 4], we need only to solve the constraint equations to first order in $\pi$ because we are only interested in cubic interactions in the Lagrangian. Solving for small fluctuations around the metric $\delta N = N-1$ to first order in $\pi$, we find that

$$\delta N = \epsilon H \pi, \tag{27}$$

$$\partial^i N_i = -\frac{\epsilon H \dot{\pi}}{c_s^2} \, , \tag{28}$$

where we have defined the slow roll parameter

$$\epsilon = -\frac{\dot{H}}{H^2} \, , \tag{29}$$

and the speed of sound as:

$$c_s^{-2} = 1 - 2\frac{M^4(t)}{\dot{H} M_{\rm Pl}^2} \, . \tag{30}$$

Plugging these back into the Lagrangian, and concentrating on only up to the next to leading terms in slow roll parameters we obtain

$$\mathcal{L} = \mathcal{L}_2 + \mathcal{L}_3, \tag{31}$$

$$\mathcal{L}_2 = a^3 \bar{M}^4 \left( \dot{\pi}^2 - \frac{c_s^2}{a^2}(\partial_i \pi)^2 + 3\epsilon H^2 \pi^2 \right), \tag{32}$$

$$\mathcal{L}_3 = a^3 \left( C_{\dot{\pi}^3} \dot{\pi}^3 + \frac{C_{\dot{\pi}(\partial\pi)^2}}{a^2} \dot{\pi} (\partial_i \pi)^2 + C_{\pi\dot{\pi}^2} \pi \dot{\pi}^2 + \frac{C_{\pi(\partial\pi)^2}}{a^2} \pi (\partial_i \pi)^2 + C_{\rm NL} \dot{\pi} \partial_i \pi \partial^i \frac{1}{\partial^2} \dot{\pi} \right), \tag{33}$$

where for brevity we defined $\bar{M}^4 \equiv \epsilon H^2 M_{\rm Pl}^2 / c_s^2 = 2M^4/(1-c_s^2)$. The cubic coefficients are

$$\begin{aligned} C_{\dot{\pi}^3} &= \bar{M}^4 \left(1 - c_s^2\right) \left(1 + \frac{2}{3} c_3\right), \\ C_{\dot{\pi}(\partial\pi)^2} &= \bar{M}^4 \left(-1 + c_s^2\right), \\ C_{\pi\dot{\pi}^2} &= \bar{M}^4 H \left(-6\epsilon + \eta - 2s + 3\epsilon c_s^2 - 2\epsilon c_3 (1 - c_s^2)\right), \\ C_{\pi(\partial\pi)^2} &= \bar{M}^4 H \left(\epsilon - \eta c_s^2\right), \\ C_{\rm NL} &= \bar{M}^4 H \left(\frac{2\epsilon}{c_s^2}\right), \end{aligned} \tag{34}$$

where we have defined the other slow roll parameters as:

$$\eta = \frac{\dot{\epsilon}}{\epsilon H} \qquad s = \frac{\dot{c}_s}{c_s H} \, . \tag{35}$$

Notice that the final term is a non-local interaction term which arises from the fact that we have written a gauge fixed Lagrangian. Obviously, all gauge invariant observables will be local. At next to leading order in slow roll parameters we have five distinct cubic operators which give rise to the five distinct shapes for the three point function.

### 3.2.2 2-point function and its tilt

In order to verify the consistency relation at first order in slow roll parameters, we need to compute the 2-point function (3) for $\zeta$ at first order in slow roll. In order to do this, we need to find the relationship between $\pi$ and $\zeta$. This is given in App. B at the non linear level. Here we just need the relationship at linear level. It is rather straightforward to realize that in Fourier space:

$$\zeta_k = -H_* \pi_k \ , \ \text{at linear level} \ . \tag{36}$$

where the $_*$ means that the quantity has to be evaluated at horizon crossing, that is when $\omega^2 = k^2/a(t_*)^2 = H(t_*)^2$. This relationship can be understood as follows. At linear level no space derivative can appear. Then, the relationship can be derived thinking of very long wavelengths. As we mentioned in sec. 2, on large scales $\zeta$ corresponds to the relative expansion between separate unperturbed FRW universes. Similarly, on large scales, $\pi$ represents the time-delay between the same separate unperturbed FRW universes. Therefore, we deduce that $\zeta = -H_*\pi$, at least at linear level, as is verified in App. B. The sign depends on our definition of $\pi$.

Now, we would need to find the wavefunction for the mode $\pi^{cl}_k(t)$ defined by the relationship $\pi_k = \pi^{cl}_k \hat{a}_k + \pi^{cl}_{-k}{}^* \hat{a}^\dagger_{-k}$. Doing this at first order in slow roll parameters is a rather tedious task, that we perform in App. C. However, one can easily find the solution in exact de Sitter from the quadratic Lagrangian (32):

$$\begin{aligned} \pi^{cl}_k(\tau) &= i\frac{1}{2\sqrt{\epsilon k^3 c_s} M_{\rm Pl}}(1 + ikc_s\tau)e^{-ikc_s\tau} \ , \\ \zeta^{cl}_k(\tau) &= -H\pi^{cl}_k \ , \end{aligned} \tag{37}$$

where we have imposed the usual Minkowski vacuum at early times. Here $\tau$ is the conformal time. The power spectrum becomes:

$$P_k = \frac{H_*^2}{4\,\epsilon\, c_{s,*}\, M_{\rm Pl}^2}\frac{1}{k^3} \ . \tag{38}$$

Here this quantity is evaluated at $t_*$ in order to minimize the error we introduced in evaluating the wavefunction in de Sitter. The dependence on $t_*$ induces an additional momentum dependence. It is convenient to parameterize it by saying that the total correlation function has the form $k^{-3+n_s}$, where

$$\begin{aligned} n_s &= \frac{d\ \log k^3 P_k}{d\ \log k} = k\frac{d}{d\,k}\log\left(\frac{H_*^4}{\dot{H}_* c_{s*} M_{\rm Pl}^2}\right) \sim \frac{1}{H_*}\frac{d}{d\,t_*}\log\left(\frac{H_*^4}{\dot{H}_* c_{s*} M_{\rm Pl}^2}\right) \\ &= 4\frac{\dot{H}_*}{H_*^2} - \frac{\ddot{H}_*}{\dot{H}_* H_*} - \frac{\dot{c}_{s*}}{c_{s*}H_*} = -2\epsilon - \eta - s \ . \end{aligned} \tag{39}$$

In App. C we explicitly verify that this is the correct result. Notice also that, as anticipated, after horizon crossing the commutator $[\zeta, \dot{\zeta}] \to 0$ exponentially fast in cosmic time.

### 3.2.3 3-point function at leading order in slow roll parameters

At leading order in slow-roll, remarkably little work or subtlety is involved, since mixing with gravity, time-dependence of coefficients, corrections to pure de Sitter wave functions, and corrections to the evolution of comoving time $\tau$ can all be ignored. At this order we have a contribution only from

two operators: $\dot{\pi}(\partial\pi)^2$ and $\dot{\pi}^3$. The calculation of non-Gaussianities in the power spectrum of fields such as $\pi$ from their interactions in the Lagrangian is a standard [1] calculation of the expectation value $\langle\pi^3\rangle$.

$$\langle\pi^3(t_0)\rangle = -i\int_{-\infty}^{t_0} dt\langle[\pi^3(t_0), \mathcal{H}_{\rm int}(t)]\rangle \tag{40}$$

where $t_0$ is some time well after all modes have exited the horizon. At leading order in interaction we have $\mathcal{H}_{\rm int} = -\mathcal{L}_{\rm int}$. We will focus first on the interaction $\mathcal{L}_{\rm int} = -2M^4\int d^3xa^3\dot{\pi}(\partial_i\pi/a)^2$. Equation (40) is evaluated for the explicit $\pi$ operators in terms of the interaction picture wavefunctions, $\pi_k = \pi_k^{cl}\hat{a}_k + \pi_{-k}^{cl\,*}\hat{a}_{-k}^\dagger$.

$$\begin{aligned}\langle\pi_{k_1}\pi_{k_2}\pi_{k_3}\rangle &= i\, C_{\dot{\pi}\partial\pi^2}(2\pi)^3\delta^3\left(\sum_i \vec{k}_i\right)\pi_{k_1}^{cl}(0)\pi_{k_2}^{cl}(0)\pi_{k_3}^{cl}(0)\\ &\quad\cdot\int_{-\infty}^{0}\frac{d\tau}{H\tau}\frac{d}{d\tau}\pi_{k_1}^{cl\,*}(\tau)\pi_{k_2}^{cl\,*}(\tau)\pi_{k_3}^{cl\,*}(\tau)\left(\vec{k}_2\cdot\vec{k}_3\right) + \text{permutations} + \text{c.c.}\end{aligned} \tag{41}$$

where the sum above includes all symmetric permutations of the three momenta. Inserting the expression for the wavefunction in eq. (37) into the above expression for the three-point function, we find

$$\begin{aligned}\langle\pi_{k_1}\pi_{k_2}\pi_{k_3}\rangle &= -iM^4(2\pi)^3\delta^3(\sum_i\vec{k}_i)\frac{1}{32c_s^3\epsilon^3HM_{\rm Pl}^6\prod_i k_i^3}\\ &\times \int_{-\infty}^{0} d\tau c_s^2k_1^2(k_2\cdot k_3)(1-ik_2c_s\tau)(1-ik_3c_s\tau)e^{i\sum_i k_ic_s\tau}\\ &+ \text{permutations} + c.c.\end{aligned} \tag{42}$$

The above integral converges if we rotate the path of integration slightly into the complex plane to pick out the interacting vacuum state in the infinite past, so that $\tau\to-\infty(1-i\epsilon)$. We can use the constraint $\sum\vec{k}_i = 0$ to simplify the above expression further. Since $\sum\vec{k}_i = 0$, it follows that $k_1\cdot k_2 = \frac{1}{2}(k_3^2-k_1^2-k_2^2)$, and permutations thereof. This makes it easy to eliminate the inner products in favor of the magnitudes of the $k_i$'s. All the cubic operators contribute to the three-point function via similar computations to the integral above. It is useful to define a basis of symmetric products of $k_i$ wavevectors. We choose the combinations

$$K_1 = k_1+k_2+k_3 \tag{43}$$

$$K_2 = (k_1k_2+k_2k_3+k_3k_1)^{1/2} \tag{44}$$

$$K_3 = (k_1k_2k_3)^{1/3} \tag{45}$$

and any symmetric polynomial in $k_i$'s can be decomposed in terms of these. Equation (42) is now easy to evaluate and express in terms of the symmetric variables $K_i$. In general, the contribution from an operator $\mathcal{O}$ in the Lagrangian which is cubic in $\pi$'s can be put in the form

$$\begin{aligned}\langle\pi_{k_1}\pi_{k_2}\pi_{k_3}\rangle &= i\int d^3x\int_{-\infty}^{0} ad\tau\sqrt{-g}\,\langle[\pi_{k_1}(\tau)\pi_{k_2}(\tau)\pi_{k_3}(\tau), C_{\mathcal{O}}\mathcal{O}]\rangle \equiv \\ &\equiv C_{\mathcal{O}}\left|\prod_{i=1}^{3}\pi_{k_i}^c(0)\right|^2(2\pi)^3\delta^3(\sum_i\vec{k}_i)\frac{A_{\mathcal{O}}(c_s,H,K_1,K_2,K_3)}{K_1^3}\end{aligned} \tag{46}$$

where in this case the $C_{\mathcal{O}}$ are $C_{\dot{\pi}^3}$ and $C_{\dot{\pi}\partial\pi^2}$ evaluated at zeroth order in slow roll parameters . Regardless of the size of $C_{\mathcal{O}}$, it is straightforward to calculate the leading-order contribution to any shape $A_{\mathcal{O}}$ by using the computational procedure described above. For $\mathcal{O} = \dot{\pi}^3$ and $\dot{\pi}(\partial_i\pi)^2$, they are

$$\begin{aligned} A_{\dot{\pi}(\partial\pi)^2} &= \left(\frac{c_s}{H}\right)\left(24K_3^6 - 8K_2^2K_3^3K_1 - 8K_2^4K_1^2 + 22K_3^3K_1^3 - 6K_2^2K_1^4 + 2K_1^6\right) \\ A_{\dot{\pi}^3} &= \left(\frac{c_s^3}{H}\right)\left(24K_3^6\right) \end{aligned} \tag{47}$$

To leading order in slow roll parameters, then, the $\pi$ 3-point function is

$$\langle\pi_{k_1}\pi_{k_2}\pi_{k_3}\rangle = \left|\prod_{i=1}^{3}\pi^{cl}_{k_i}(0)\right|^2 (2\pi)^3\delta^3(\sum_i \vec{k}_i)\frac{2M^4\left(-A_{\dot{\pi}\partial\pi^2} + (1+2c_3/3)A_{\dot{\pi}^3}\right)}{K_1^3} \tag{48}$$

To find the 3-point function for $\zeta$ ($\zeta$ is constant outside of the horizon, and therefore it is the relevant quantity for observation), we need the relation between $\pi$ and $\zeta$. This can be found performing the diffeomorphism that connects the $\zeta$-gauge (6) and the $\pi$-gauge (24). This is given in App. B [1]:

$$\zeta = -H\pi + H\pi\dot{\pi} + \tfrac{1}{2}\dot{H}\pi^2 + \alpha \tag{49}$$

$$4\alpha = \frac{1}{a^2}\left(-\partial_i\pi\partial_i\pi + \partial^{-2}\partial_i\partial_j(\partial_i\pi\partial_j\pi)\right) \tag{50}$$

The $\alpha$ term is proportional to spatial derivatives of $\pi$, so it vanishes outside the horizon and will not contribute in the expression for the three-point function. The $\frac{1}{2}\dot{H}\pi^2$ in $\zeta$ contributes to the $\zeta$ three-point function through $\langle\zeta^3\rangle \supset \langle H^2\dot{H}\pi^4\rangle$, which is sub-leading in slow roll expansion with respect to the terms we are keeping here. Finally, the $H\pi\dot{\pi}$ in $\zeta$ contributes through $\langle\zeta^3\rangle \supset \langle H^3\pi^3\dot{\pi}\rangle$, but this also vanishes at this order in slow-roll since in exact de Sitter space $\dot{\pi}$ vanishes outside the horizon. The full three-point function for $\zeta$ to leading order in $\epsilon$ is therefore

$$\begin{aligned} \langle\zeta_{k_1}\zeta_{k_2}\zeta_{k_3}\rangle &= -H^3\langle\pi_{k_1}\pi_{k_2}\pi_{k_3}\rangle = -\pi^3\delta^3(\sum_i\vec{k}_i)\frac{H^3M^4\left(-A_{\dot{\pi}\partial\pi^2} + (1+2c_3/3)A_{\dot{\pi}^3}\right)}{4c_s^3M_{\rm Pl}^6\epsilon^3K_3^9K_1^3} \\ &= \pi^3\delta^3(\sum_i\vec{k}_i)H^3\dot{H}(1-c_s^2)\frac{\left(-A_{\dot{\pi}\partial\pi^2} + (1+2c_3/3)A_{\dot{\pi}^3}\right)}{8c_s^5M_{\rm Pl}^4\epsilon^3K_3^9K_1^3} \end{aligned} \tag{51}$$

where in the last step we have used $M^4 = \dot{H}M_{\rm Pl}^2(1-1/c_s^2)/2$. In App. D we show using our generic effective theory (16) that $\zeta$ is constant outside of the horizon at fully non linear level.

#### 3.2.4 3-point function at next to leading order in slow roll parameters

At next to leading order in slow-roll, the computation of the non-Gaussianities involves considerably more work, and much of the computational advantage of using the $\pi$ field instead of the $\zeta$ field is lost. This is understandable, as the Goldstone boson is advantageous for describing the UV physics, while here we are also considering some IR effects. At any order in slow roll, the contribution from an operator $\mathcal{O}$ is still exactly given by the general expression (46). The calculation of $C_{\mathcal{O}}$, $\pi^{cl}_k(0)$ and $A_{\mathcal{O}}/K_1^3$ should be carried out at the required order, but they are all independent of each other.

We have just calculated the coefficients $C_{\mathcal{O}}$ at the next leading order in slow roll parameters. The next to leading order corrections involve three new operators that were not present at leading order.

For these three operators, the contribution to the non-Gaussianities is actually quite easy. Their coefficients are already suppressed by slow-roll, so we can simply use the leading order contributions to $A_{\mathcal{O}}$, and be done. The operators $\dot{\pi}^3$ and $\dot{\pi}(\partial\pi)^2$ that were present at leading order are more complicated. In addition to the slow-roll corrections to $C_{\dot{\pi}^3}$ and $C_{\dot{\pi}(\partial\pi)^2}$, there are several corrections inside the integral of equation (46). Such corrections have been treated before[4, 23], but we will not need them in our calculation. We can write the integral to evaluate symbolically as

$$\begin{aligned}\langle\pi_{k_1}(0)\pi_{k_2}(0)\pi_{k_3}(0)\rangle &= i\int d^3x\int_{-\infty}^{0}[\text{ MEASURE }]\,[\text{ TIME-DEPENDENCE }]\\ &\times\quad[\text{ DERIVATIVES }]\,[\text{ MODES }]+\text{c.c.}+\text{symm.}\quad . \end{aligned}\tag{52}$$

It is not very instructive to write the full result, because we cannot put it in closed form in any case. Instead, we turn directly to verify the consistency relation, as we can do it without many complication. In fact, there is a nice trick that we can use in the squeezed limit to avoid these complications. To understand why this is the case, let us first ask which operators contribute in the squeezed limit if we work in the $\zeta$ gauge instead of the $\pi$ gauge. A key insight is that $\zeta$ does not evolve outsize the horizon, and so operators like $\dot{\zeta}^3$ and $\dot{\zeta}(\partial\zeta)^2$, with a derivative on each $\zeta$, do not contribute in the squeezed limit. If we imagine doing the calculation directly for $\zeta$, we see that the contribution from the operator $\dot{\zeta}^3$ will be proportional to the derivative of $\zeta_{k_L}$, and therefore will be negligible in the squeezed limit. However, the same reasoning does not apply to $\pi=-\zeta/H+...$, which evolves outside the horizon, and therefore the operators $\dot{\pi}^3$ and $\dot{\pi}(\partial\pi)^2$ still contribute in the squeezed limit. However, the interaction picture wavefunction, which follows the linear equation of motion, is *almost* constant outside the horizon. In particular, using that $\zeta^{cl}=-H\pi^{cl}$, we can write:

$$\dot{\pi}^{cl}=-\frac{\dot{\zeta}^{cl}}{H}+H\epsilon\pi^{cl}\tag{53}$$

We emphasize that this relation is true for the classical solutions, but not for the $\pi$ and $\zeta$ operators in general, and that one should only use the above substitution under the integral in the computation of the 3-point functions (46). But this is enough for us. We have to compute corrections only to terms that were already present at leading order in slow roll, $\dot{\pi}^3$ and $\dot{\pi}(\partial\pi)^2$. Focussing on the latter of these for the moment, we make the replacement

$$\dot{\pi}^{cl}(\partial\pi^{cl})^2\rightarrow-\frac{1}{H}\dot{\zeta}^{cl}(\partial\pi^{cl})^2+H\epsilon\pi^{cl}(\partial\pi^{cl})^2\tag{54}$$

The second term on the RHS is now suppressed by $\epsilon$, so its shape can be calculated without any slow-roll corrections, as we shall soon do. The first term is clearly going to be sub-leading in the squeezed triangle limit, because there is one derivative on each field, and therefore we expect a suppression of order $k_L/k_S$ with respect to the leading behavior. However, since, in the actual computation, this term is under an integral over $\tau$, it might not be obvious to the reader that this contribution really vanishes. This last fact can be explicitly verified, recovering the $k$ dependence of the contribution of the first term on the RHS of (54) in the squeezed limit. Let us see how we can do this. We have to evaluate the following integral:

$$\begin{aligned}\langle\pi_{k_1}\pi_{k_2}\pi_{k_3}\rangle &= -i\,C_{\dot{\pi}\partial\pi^2}(2\pi)^3\delta^3(\sum_i\vec{k}_i)\pi^{cl}_{k_1}(0)\pi^{cl}_{k_2}(0)\pi^{cl}_{k_3}(0)\\ &\cdot\int_{-\infty}^{0}\frac{d\tau}{H^2\tau}\frac{d}{d\tau}\zeta^{cl\,*}_{k_1}(\tau)\;\pi^{cl\,*}_{k_2}(\tau)\pi^{cl\,*}_{k_3}(\tau)\left(\vec{k}_2\cdot\vec{k}_3\right)\;+\;\text{c.c.}\;+\;\text{symm.}\quad .\end{aligned}\tag{55}$$

where we have contracted $\pi_{k_1}$ with $\dot{\zeta}_k$ as an illustrative example. We can divide the integral over conformal time into two regions: "early times", when $k_S \gg k_L \gtrsim aH$ and the physical wavelengths are inside the horizon; "late times", when $k_L \ll aH$ and the long physical wavelength is outside the horizon. Now, the contribution from early times is negligible due to the rapid oscillations from the exponential factor $e^{ik_S c_s \tau}$ [5]. In the remainder of the integral, $k_L$ can safely be taken as small as we like. In this limit, then the Hankel function of the field which has the long wavelength mode can be expanded in the small argument limit. It is clear that this expansion will bring powers of $k_L$ in the numerator, making the contribution of this term negligible in the squeezed limit[6]. But let us explicitly see how this works considering the particular case where $k_1 = k_L$, and $k_2 \simeq k_3 = k_S$. In this case we can expand the wave function of $\zeta$, as done in App. C.1. Considering only the parametric dependence on momenta, the first term from equation (55) contributes as

$$
\begin{aligned}
\langle \pi_{k_L}\pi_{k_s}\pi_{k_s}\rangle &\propto \pi^{cl}_{k_L}(0)\pi^{cl}_{k_s}(0)\pi^{cl}_{k_s}(0)\cdot\int_{-\infty}^{0}\frac{d\tau}{\tau}\frac{1}{\tau}\tau^2 k_L^{1/2}k_S^2\pi^{cl*}_{k_s}(\tau)\pi^{cl*}_{k_s}(\tau) \\
&\propto \frac{1}{k_L^{3/2}}\frac{1}{k_S^3}\cdot\int_{-\infty}^{0}\frac{d\tau}{\tau}\frac{1}{\tau}\tau^2 k_L^{1/2}\left(k_S(-\tau)^{3/2}H_\nu^{(1)}(-c_s k_S\tau(1+s))\right)^2 \\
&\propto \frac{1}{k_L^3}\frac{1}{k_S^3}\left(\frac{k_L}{k_S}\right)^2\int_{-\infty}^{0}dy\ y^2(H_\nu^{(1)}(-y))^2\ . \qquad (56)
\end{aligned}
$$

where in the first passage we have used the long wavelength limit of $\dot{\zeta}$ given in App. C.1, in the second we have used the explicit form for the wave functions at first order in slow roll parameters given in (131), and in the third we have changed variable of integration to $y = c_s k_S \tau(1+s)$. We see that the remaining integral is just a numerical factor, and the $k$ dependence of the contribution is suppressed with respect to the leading one in the consistency relation by a factor of $(k_L/k_S)^2$. It is clear that the same arguments apply to the other terms of eq.(55). We avoid showing the very similar argument for the operator $\dot{\pi}^3$ except there is an extra factor of 3, since $\pi_{k_L}$ can be put on any of the three $\dot{\pi}$'s. Notice that this works out so that the terms in $c_3$ coming from $(1/N^2-1)^3$ do not contribute in the squeezed limit. This is very important for the consistency relation to be true, as no term from $(1/N^2-1)^3$ contributes to the two point function, and therefore to its tilt.

Applying this argument, we find that the $\pi$ three-point function in the squeezed limit ($k_1 \to 0$) at next to leading order is

$$
\begin{aligned}
\langle \pi_{k_1}\pi_{k_2}\pi_{k_3}\rangle &= \left|\prod_{i=1}^{3}\pi^{cl}_{k_i}(0)\right|^2 (2\pi)^3\delta^3(\sum_i \vec{k}_i) \\
&\times\ \frac{(C_{\dot{\pi}^3}3H\epsilon + C_{\pi\dot{\pi}^2})A_{\pi\dot{\pi}^2} + (C_{\dot{\pi}(\partial\pi)^2}H\epsilon + C_{\pi(\partial\pi)^2})A_{\pi\partial\pi^2}}{K_1^3} \qquad (57)
\end{aligned}
$$

The shapes $A_{\pi(\dot{\pi})^2}$ and $A_{\pi(\partial\pi)^2}$ can be calculated using the method described in section 3.2.3. We obtain

$$
\begin{aligned}
A_{\pi(\partial\pi)^2} &= \left(\frac{c_s}{H^2}\right)\left(4K_2^2K_3^3K_1 + 4K_2^4K_1^2 - 2K_3^3K_1^3 - 6K_2^2K_1^4 + 2K_1^6\right) \qquad (58) \\
A_{\pi\dot{\pi}^2} &= \left(\frac{c_s^3}{H^2}\right)\left(4K_2^2K_3^3K_1 + 4K_2^4K_1^2 - 8K_3^3K_1^3\right)
\end{aligned}
$$

[5] In practice, the oscillatory damping is even more obvious since one analytically continues into the complex plane where it becomes exponential damping.

[6] This is the step of the proof that fails for the term $\dot{\pi}(\partial\pi)^2$ but holds for $\dot{\zeta}(\partial\pi)^2$.

At next to leading order, $\zeta = -H\pi + H\pi\dot\pi + \frac{1}{2}\dot H\pi^2 + 4\alpha$. Then, in Fourier space, $\zeta$ is of the form $\zeta_k = -H\pi_k + H(\pi * \dot\pi)_k + \frac{1}{2}\dot H(\pi*\pi)_k$, where $(\pi*\pi)_k = \int d^3q\ \pi_{k-q}\pi_q/(2\pi)^3$ is a convolution. As before, $\dot\pi$ outside the horizon can be replaced by $H\epsilon\pi$ and the terms contained in $\alpha$ are irrelevant, so outside the horizon we can take $\zeta_k = -H\pi_k - \frac{1}{2}\dot H(\pi*\pi)_k$. Considering the contribution from the field redefinition, the full three-point function for $\zeta$ in the limit $k_1 = k_L \to 0$ is

$$\begin{aligned}
\langle\zeta_{k_1}\zeta_{k_2}\zeta_{k_3}\rangle &= -H^3\langle\pi_{k_1}\pi_{k_2}\pi_{k_3}\rangle - \frac{\dot H}{H^2}(2\pi)^3\delta^3(\sum_i \vec k_i)\left(P_{k_1}P_{k_2}+P_{k_2}P_{k_3}+P_{k_1}P_{k_3}\right) \\
&\simeq -(2\pi)^3\delta^3(\sum_i \vec k_i)P_{k_S}P_{k_L} \\
&\quad\times\left(\left|\pi^{cl}_{k_S}(0)\right|^2\frac{(C_{\dot\pi^3}3H\epsilon + C_{\pi\dot\pi^2})A_{\pi\dot\pi^2} + (C_{\dot\pi(\partial\pi)^2}H\epsilon + C_{\pi(\partial\pi)^2})A_{\pi(\partial\pi)^2}}{8k_S^3 H} - 2\epsilon\right)
\end{aligned} \tag{59}$$

Here $P_k$ is the power spectrum defined in eq.(3). In the first line, the second term in the right hand side comes from the field redefinition. In the second line we have taken the squeezed limit, and used that $P_{k_S} \ll P_{k_L}$.

It is straightforward to check that the last term above in parentheses is equal at next to leading order to $n_s$. In the squeezed limit we have,

$$A_{\pi(\partial\pi)^2} = \frac{48c_s k_S^6}{H^2}\ , \qquad A_{\pi\dot\pi^2} = \frac{16c_s^3k_S^6}{H^2}\ . \tag{60}$$

From equation (37),

$$|\pi^{cl}_{k_S}(0)|^2 = \frac{1}{4\epsilon k_S^3 c_s M_{\rm Pl}^2}\ , \tag{61}$$

and from equation (34) we have

$$C_{\dot\pi^3}3H\epsilon + C_{\pi\dot\pi^2} = M_{\rm Pl}^2H^3\epsilon(-2s-3\epsilon+\eta)/c_s^2\ , \qquad C_{\dot\pi(\partial\pi)^2}H\epsilon + C_{\pi(\partial\pi)^2} = M_{\rm Pl}^2H^3\epsilon(\epsilon-\eta)\ . \tag{62}$$

Substituting these expressions in (59), we find

$$\langle\zeta_{k_1}\zeta_{k_2}\zeta_{k_3}\rangle \simeq -(2\pi)^3\delta^3(\sum_i \vec k_i)P_{k_S}P_{k_L}(-2\epsilon-s-\eta) = -(2\pi)^3\delta^3(\sum_i \vec k_i)P_{k_S}P_{k_L}n_s \tag{63}$$

which is exactly of the form of eq.(2) if we identify $k_1 = k_L \to 0$ and $k_2 \simeq k_3 = k_S$ [7] . This ends the verification of the consistency relation at first order in slow roll parameters for the particular case we have considered in this section.

### 3.3 Verification for the Operator $\delta N\delta E^i{}_i$

We now turn to the verification of the consistency relation in the case where the operators which involve the extrinsic curvature are important. Since these terms are higher derivative, in general they do not give rise to the leading contribution. However this is not necessarily the case if we are

[7] The 3-point function produced by a scalar field with Lagrangian of the form $P(-(\partial\phi)^2,\phi)$ had already been studied in [4], where it was originally found that the consistency relation was violated by terms of the form $\epsilon/c_s^2$. However, after the publication of this paper, the authors of [4] revised their calculations and found some algebra mistakes. The new results of [4] agree with ours completely.

near de Sitter. As we explain more in detail in [17], the leading gradient term to the $\pi$ field is fixed by the symmetries of FRW to be equal to $M_{\rm Pl}^2\dot{H}(\partial_i\pi)^2$. This term goes to zero as $\dot{H}$ vanishes, and this allows higher derivative terms to become the leading ones. This is the case of the terms in $\delta E^{i\,2}_{\ i}$ and $\delta E^i{}_j\delta E^j{}_i$, which, upon reinsertion of the $\pi$, give rise to:

$$-\frac{1}{2}d_2 M^2\,\delta E^i{}_i{}^2-\frac{1}{2}d_3\,M^2\,\delta E^{ij}\delta E_{ij}\ \to -\frac{1}{2}\bar{d}M^2\,\frac{1}{a^4}(\partial_i^2\pi)^2\ , \tag{64}$$

where $\bar{d}=d_2+d_3$. The situation for the operator $\delta N\delta E^i{}_i$ is slightly different. Upon reintroduction of the $\pi$, this term gives a non trivial contribution to the action only either through its mixing with gravity, or because of the presence of the Hubble constant. Neglecting metric fluctuations and concentrating only on the $\pi$ terms, we obtain:

$$-\frac{d_1}{2}M^3\dot{\pi}\frac{1}{a^2}\partial^2\pi\to\frac{d_1}{2}M^3\frac{1}{2a^2}\frac{d}{dt}(\partial_i\pi)^2\to-\frac{Hd_1M^3}{4}\frac{1}{a^2}(\partial_i\pi)^2\ , \tag{65}$$

where we have performed an integration by parts. Since this two derivative gradient term is suppressed by $H$ it is in general negligible unless $\dot{H}$ is very small. This is the reason why, in the former section, we completely neglected these terms. However, even though by a small amount, these operator do contribute to the 2-point function and to its tilt, and therefore, if the consistency relation is true, they must give a contribution to the 3-point function in the squeezed limit. This in principle should be checked [8]. As it should be clear from the study of the former section, the full study of the three point function in the squeezed limit in the case we include all these operators is clearly very long and tedious. Furthermore, it is true that these new operators become important only if we are close to de Sitter. For these reasons, in the rest of this section we decide to make some simplifying assumptions. First of all, we restrict to de Sitter background, so that we make the importance of these operators as large as possible. Second, for each operator, we decide to restrict ourself at verifying the consistency relation for the minimal case which is still non-trivial, setting all the unnecessary operators to zero. In summary, we consider two separate cases: one where we set to zero the operators $\delta E^i{}_i{}^2$ and $\delta E^i{}_j\delta E^j{}_i$, and we keep the operators $(1/N^2-1)^2$ and $\delta N\delta E^i{}_i$; and the other where we set to zero $\delta N\delta E^i{}_i$ and $\delta E^i{}_i{}^2$ and we keep $(1/N^2-1)^2$ and $\delta E^i{}_j\delta E^j{}_i$ [9]. We then start by considering the following action:

$$S_{\rm matter}=\int d^4x\,\sqrt{-g}\left[\frac{M^4(t)}{2}\left(\frac{1}{N^2}-1\right)^2+\frac{d_1(t)}{2}M^3(t)\delta N\delta E^i_i\right]\ . \tag{66}$$

Notice that we have set to zero the operator $(1/N^2-1)^3$ because it is not strictly necessary for a non trivial check of the consistency relation. In the squeezed limit, the consistency relation involves the deviation from scale invariance of the two point function. Even though we are in de Sitter, still we obtain a scale dependence from the time dependence of the coefficients in (66), which allows us

[8]At this point one could wonder of the particular regime in which we are close to de Sitter and in which the coefficients of the extrinsic curvature operators we consider are tuned to be zero. Then other higher derivative terms can become important. However, as we show in [17], when this is the case, the theory becomes strongly coupled in the infrared, and our effective field theory description breaks down. We therefore conclude that it is enough to restrict ourself to study the operators we consider.

[9]The case where we keep $(1/N^2-1)^2$ and $\delta E^i{}_i{}^2$ and set the other operators to zero would be very similar to the second case we consider, and therefore we avoid to explicitly perform the calculation for this case.

to have a non trivial check. To this purpose, we keep only the first time derivative and drop higher ones.

As we did in the former section, we need to find the effective action for the relevant scalar degree of freedom. This involves reinserting the $\pi$, writing down the constraint equations, gauge fixing and solving them in terms of $\delta N$ and $N_i$, and finally plugging back into the gauge fixed action. This part proceeds as in the former section, with just the algebra being a bit more complicated.

Reinsertion of the $\pi$ in the operators $\delta E^i{}_i$ leads to (see App. E for a detailed derivation):

$$\begin{aligned}\delta E^i_i \quad \rightarrow \quad & \delta E^i_i - 3H\dot{\pi} + 3H\dot{\pi}^2 - \tfrac{1}{2}\partial_t(2\partial_i\pi N^i + (\partial\pi)^2) \\ & -\partial^2\pi - \partial_i(2\delta N\partial^i\pi + 3N^i\dot{\pi} - 2\partial^i\pi\dot{\pi})\,. \end{aligned} \tag{67}$$

The constrained equation can be solved in the spatially flat gauge we use (eq. (24)) to give:

$$\delta N = \frac{d_1(t)M^3(t)}{d_1(t)M^3(t) + 4HM_{\rm Pl}^2}\,\dot{\pi} \tag{68}$$

$$\begin{aligned}\partial_i N^i = & -\frac{24d_1(t)H^2M_{\rm Pl}^2M(t)^3 + 32HM_{\rm Pl}^2M(t)^4 + 3d_1(t)^2HM(t)^6}{\left(4HM_{\rm Pl}^2 + d_1(t)M^3(t)\right)^2}\,\dot{\pi} \\ & -\frac{d_1(t)M^3(t)}{4HM_{\rm Pl}^2 + d_1(t)M^3(t)}\,a^{-2}\partial^2\pi \end{aligned} \tag{69}$$

Plugging back in (66) we obtain the final action.

### 3.3.1 2-point function and its tilt

At quadratic level, the action is:

$$S_2 = \frac{1}{2}\int d^3x d\tau f^2(\pi'^2 - c_s^2(\partial_i\pi)^2) \tag{70}$$

where we have directly passed to conformal time. $f$ and $c_s$ are defined by:

$$f^2\tau^2 = M_{\rm Pl}^2\frac{64M^4M_{\rm Pl}^2 + 6d_1M^3(8HM_{\rm Pl}^2 + d_1M^3)}{(4HM_{\rm Pl}^2 + d_1M^3)^2}\,, \tag{71}$$

$$c_s^2 = -\frac{d_1M^3(4HM_{\rm Pl}^2 + d_1M^3) + 4M_{\rm Pl}^2\partial_t(d_1M^3)}{\left(32M^4M_{\rm Pl}^2 + 3d_1M^3(8HM_{\rm Pl}^2 + d_1M^3)\right)}\,, \tag{72}$$

where for brevity we have stopped explicitly showing that $M$ and $d_1$ depend on time. The quadratic action is the one of a scalar field with a speed of sound different from one. Using the result of the former section, we can quickly compute the power spectrum and its tilt:

$$P_k = \frac{H^2}{2f^2\tau^2c_s^3}\frac{1}{k^3}\,, \tag{73}$$

$$\begin{aligned} n_s \simeq \frac{d\log k^3P_k}{d\log k} = & \frac{16M_{\rm Pl}^2\partial_tM_*^4}{H\left(32M_{\rm Pl}^2M_*^4 + 3d_{1*}M_*^3(8HM_{\rm Pl}^2 + d_{1*}M_*^3)\right)} \\ & -M_{\rm Pl}^2\frac{\left(32M_*^4(6HM_{\rm Pl}^2 + d_{1*}M_*^3) + 6Hd_{1*}M_*^3(16HM_{\rm Pl}^2 + 3d_{1*}M_*^3)\right)\partial_t(d_1M^3)_*}{Hd_{1*}M_*^3(4HM_{\rm Pl}^2 + d_{1*}M_*^3)\left(32M_{\rm Pl}^2M_*^4 + 3d_{1*}M_*^3(8HM_{\rm Pl}^2 + d_{1*}M_*^3)\right)} \end{aligned} \tag{74}$$

where the subscript $_*$ stands for evaluation at horizon crossing $\omega \simeq H$.

### 3.3.2 3-point function in the squeezed limit

In the de Sitter limit the computation of the 3-point function in the squeezed limit gets largely simplified. This comes from the fact that now (see App. B)

$$\zeta = -H\pi + H\dot{\pi}\pi \,. \tag{75}$$

This means that outside the horizon $\pi$ becomes constant. This simplifies the calculation in two different ways. On the one hand this means that there are no terms in the 3-point function of $\zeta$ which come from the non linear relationship between $\zeta$ and $\pi$. On the other hand, by reproducing the argument we made in sec. 3.2.4, the interaction operators with a derivative acting on each of the $\pi$'s do not contribute in the squeezed limit, and therefore can be neglected. Concentrating on the interaction Lagrangian with at least one $\pi$ without derivatives, we are left with:

$$S_3 \quad \supset \quad \int d^3x dt \left[ aC_{\pi\dot{\pi}\partial^2\pi}\pi\dot{\pi}\partial^2\pi + a^3 C_{\pi\dot{\pi}^2}\pi\dot{\pi}^2 \right] \tag{76}$$

where

$$\begin{aligned} C_{\pi\dot{\pi}^2} &= 16H^2M_{\rm Pl}^4\frac{(4HM_{\rm Pl}^2+d_1M^3)2\partial_tM^4+2(3H^2M_{\rm Pl}^2-2M^4)\partial_t(d_1M^3)}{(4HM_{\rm Pl}^2+d_1M^3)^3}\,, \\ C_{\pi\dot{\pi}\partial^2\pi} &= 8H^2M_{\rm Pl}^4\frac{\partial_t(d_1M^3)}{(4HM_{\rm Pl}^2+d_1M^3)^2}\,. \end{aligned} \tag{77}$$

In the squeezed limit, we can put the first term in a form we are more familiar with by an integration by parts:

$$\begin{aligned} a\, C_{\pi\dot{\pi}\partial^2\pi}(t)\pi\dot{\pi}\partial^2\pi &= -a\, C_{\pi\dot{\pi}\partial^2\pi}(t)(\partial\pi\dot{\pi}\partial\pi + \pi\partial\dot{\pi}\partial\pi) \\ &= -a\, C_{\pi\dot{\pi}\partial^2\pi}(t)(\pi\tfrac{1}{2}\partial_t(\partial\pi)^2) \\ &= \tfrac{1}{2}a\, \pi(\partial\pi)^2(HC_{\pi\dot{\pi}\partial^2\pi}(t) + \dot{C}_{\pi\dot{\pi}\partial^2\pi})\,, \end{aligned} \tag{78}$$

where in the third passage we have neglected operators with one derivative on each $\pi$. So in the squeezed limit we can consider the operator $\pi(\partial\pi)^2$ instead of $\pi\dot{\pi}\partial^2\pi$ upon the definition

$$C_{\pi(\partial\pi)^2} \equiv \frac{1}{2}(HC_{\pi\dot{\pi}\partial^2\pi} + \dot{C}_{\pi\dot{\pi}\partial^2\pi})\,. \tag{79}$$

At this point it becomes immediate to obtain the result for the $\zeta$ 3-point function using (46) in the squeezed limit $k_1 = k_L \ll k_S = k_2 \simeq k_3$:

$$\begin{aligned} \langle\zeta_{k_1}\zeta_{k_2}\zeta_{k_3}\rangle &= -H^3\langle\pi_{k_1}\pi_{k_2}\pi_{k_3}\rangle \\ &\simeq -(2\pi)^3\delta^3(\sum_i\vec{k}_i)P_{k_S}P_{k_L}\left(\left|\pi^{cl}_{k_S}(0)\right|^2\frac{C_{\pi\dot{\pi}^2}A_{\pi\dot{\pi}^2}+C_{\pi(\partial\pi)^2}A_{\pi(\partial\pi)^2}}{8k_S^3H}\right) \end{aligned}$$

We have already computed $A_{\pi(\partial\pi)^2}$ and $A_{\pi\dot{\pi}^2}$ in eq.(58). Using this and the fact that

$$|\pi_{k_S}(0)|^2 = \frac{1}{2k_S^3c_s^3(f\tau)^2} \tag{80}$$

upon substitution of eq.(71) we find a complicated expression which is nothing but:

$$\langle\zeta_{k_1}\zeta_{k_2}\zeta_{k_3}\rangle \simeq -(2\pi)^3\delta^3(\sum_i\vec{k}_i)P_{k_S}P_{k_L}n_s\,, \quad k_1 = k_L \ll k_S = k_2 \simeq k_3\,, \tag{81}$$

with $n_s$ given by eq.(74). This verifies that the consistency relation holds also in this case.

### 3.4 Verification for the Operator $\delta E^i{}_j \delta E^j{}_i$: the Ghost Condensate

Now we turn to the final verification of the consistency relation, and we concentrate on the following action:

$$S = \int d^4x \, \sqrt{-g} \left[ \frac{M^4(t)}{2} \left( \frac{1}{N^2} - 1 \right)^2 - \frac{d_3(t)}{2} M^2(t) \delta E_{ij} \delta E^{ij} \right] \, . \tag{82}$$

in de Sitter space. Notice that this Lagrangian is very similar to the Lagrangian of the Ghost Condensate [24, 2]. As explained in the former section, the $\delta E^i{}_j \delta E^j{}_i$ term introduces a spatial kinetic term for $\pi$ of the form

$$(\partial_i \partial_j \pi)^2 \, . \tag{83}$$

Already at the level of solving for the 2-point function, this fact makes things rather more complicated, as we will see. Furthermore, the mixing with gravity generates a term of the form [24]

$$(\partial_i \pi)^2 \, , \tag{84}$$

which makes the calculation of the 2-point function even more complicated and probably undoable. However, there is a simplifying limit we can use. This fact is carefully explained in [17], and we refer to there for a detailed discussion. Here instead we briefly enunciate the simplifying limit. In an inflationary background, we need to study the perturbations as they redshift from some ultraviolet scale $\Lambda$ to an infrared scale which is given by the Hubble constant $H$, beneath which the $\zeta$ perturbation does not evolve anymore. Now, the terms which come from the mixing with gravity (for example a term like $\delta N \dot{\pi}$, which, upon substitution of the solution to the constraint equation $\delta N(\pi)$ becomes an operator expressible only in terms of $\pi$) are less important than the pure $\pi$ terms at energies larger than a demixing scale $\Lambda_{\rm Mix} \simeq d_3 M^3/M_{\rm Pl}^2$ [10]. Now, if $H \gg \Lambda_{\rm Mix}$, then the contribution from the mixing operators is always parametrically suppressed. In this limit, we can therefore reintroduce the $\pi$ in the action (82) and then put to zero all the metric fluctuations. The result we will obtain for the 3-point function will be wrong by a small amount parameterized by $\Lambda_{\rm Mix}/H \ll 1$. Therefore, reinserting the $\pi$ in (82) and setting the metric fluctuations to zero, we obtain

$$S = \int d^4x \sqrt{-g} \left[ 2M^4(t+\pi)\dot{\pi}^2 - \frac{1}{2} d_3(t+\pi) M^2(t+\pi) \frac{1}{a^4} (\partial_i \partial_j \pi)^2 \right] \tag{85}$$

Here we have neglected some higher derivative terms that do not contribute in the squeezed limit because, as we explained in the former section, $\pi$ goes to constant outside of the horizon in the de Sitter limit. We have also neglected a term suppressed by $H/M \ll H/\Lambda_{\rm Mix}$. The action can be written as

$$S = \int d^4x \sqrt{-g} \left[ 2M^4 \dot{\pi}^2 - \frac{1}{2} d_3 M^2 \frac{1}{a^4} (\partial_i \partial_j \pi)^2 - \partial_t (d_3 M^2) \frac{1}{2a^4} \pi (\partial_i \partial_j \pi)^2 + 8\dot{M} M^3 \pi \dot{\pi}^2 \right] \, . \tag{86}$$

[10] This is a consequence of the Goldstone equivalence theorem applied to the field $\pi$ which non linearly realizes the time diffs.

### 3.4.1 2-point function and its tilt

Solving the wave equation we find [2]:

$$\pi_k^{cl}(\tau) = -\sqrt{\frac{\pi}{8}}\frac{H}{2M^2}|\tau|^{3/2} H^{(1)}_{\frac{3}{4}}\left(\frac{Hk^2\sqrt{d_3}M}{4M^2}\tau^2\right) \ , \tag{87}$$

where

$$\pi_k(\tau) = \pi_k^{cl}(\tau)a_k + \pi_{-k}^{cl\,*}(\tau)a^\dagger_{-k}. \tag{88}$$

The spectrum of $\zeta$ is

$$P_k \quad = \quad \frac{\sqrt{2}\pi H^{5/2}}{\Gamma^2(1/4)k^3}\frac{1}{M_*\left(d_{3*}M_*^2\right)^{3/4}} \ , \tag{89}$$

and the tilt is

$$n_s \simeq \frac{1}{H_*}\frac{d}{dt_*}\left(-\log(M_*) - \frac{3}{4}\log(d_{3*}M_*^2)\right) = -\frac{\dot{M}_*}{H_* M_*} - \frac{3\partial_t(d_{3*}M_*^2)}{4H_* d_{3*}M_*^2} \ . \tag{90}$$

### 3.4.2 3-point function in the squeezed limit

From eq.(86), we see that we have to compute the contribution of two operators in the squeezed limit ($k_1 = k_L \to 0$). Let us start with $\pi(\partial_i\partial_j\pi)^2$:

$$\begin{aligned}
\langle \pi_{k_1}\pi_{k_2}\pi_{k_3}\rangle_{\pi(\partial_i\partial_j\pi)^2} \quad &= \quad -i\partial_t(d_3M^2)(2\pi)^3\delta^3\big(\sum_i k_i\big)\pi^{cl}_{k_1}(0)\pi^{cl}_{k_2}(0)\pi^{cl}_{k_3}(0) \\
&\quad \times\int d\tau\, \pi^{cl*}_{k_1}(\tau)\pi^{cl*}_{k_2}(\tau)\pi^{cl*}_{k_3}(\tau)(k_2\cdot k_3)^2 + c.c. \\
&= \quad -i\partial_t(d_3M^2)(2\pi)^3\delta^3\big(\sum_i k_i\big)\frac{-iH^{3/4}\Gamma^3(3/4)}{(d_3M^2)^{9/8}(\sqrt{2}\pi M)^{3/2}(k_1k_2k_3)^{3/2}} \\
&\quad \times\int d\tau\, \pi^{cl*}_{k_1}(\tau)\pi^{cl*}_{k_2}(\tau)\pi^{cl*}_{k_3}(\tau)(k_2\cdot k_3)^2 + c.c. \\
&\simeq \quad -i\partial_t(d_3M^2)(2\pi)^3\delta^3\big(\sum_i k_i\big)\frac{-iH^{3/4}\Gamma^3(3/4)k_S}{(d_3M^2)^{9/8}(\sqrt{2}\pi M)^{3/2}k_L^{3/2}} \\
&\quad \times\int d\tau\left(\pi^{cl*}_{k_S}(\tau)\right)^2\left(-i\frac{1}{2M^2}\left(\frac{8HM^3}{d_3^{3/2}}\right)^{1/4}\frac{\Gamma(3/4)}{k_L^{3/2}\sqrt{\pi}}\right) + c.c. \\
&= \quad i\partial_t(d_3M^2)(2\pi)^3\delta^3\big(\sum_i k_i\big)\frac{H\Gamma^4(3/4)k_S}{(d_3M^2)^{3/2}2M^2\pi^2k_L^3}\int d\tau\left(\pi^{cl*}_{k_S}(\tau)\right)^2 + c.c. \ .
\end{aligned} \tag{91}$$

where in the third passage we have used the small argument expansion of the wavefunction (87). This approximation is justified for the same reason as we argued in sec. 3.2.4. Now the integral is

$$\begin{aligned} I_{\pi(\partial_i\partial_j\pi)^2} &= \int d\tau \left(\pi^{cl*}_{k_S}(\tau)\right)^2 \\ &= \frac{\pi}{32}\frac{H^2}{M^4}\left[\int d\tau\,\tau^3\left(H^{(1)}_{3/4}\left(\frac{Hk_S^2\sqrt{d_3}M}{4M^2}\tau^2\right)\right)^2\right]^* \\ &= \frac{\pi}{32}\frac{H^2}{M^4}\frac{16M^2}{d_3H^2k_S^4}\frac{1}{2}\left[\int dx\,x\left(H^{(1)}_{3/4}(x)\right)^2\right]^* \\ &= \frac{\pi}{4}\frac{1}{d_3M^2k_S^4}\left(-\frac{3}{2\pi}(1-i)\right) \\ &= -\frac{3}{8}\frac{1}{d_3M^2k_S^4}(1-i) \end{aligned} \tag{92}$$

via the substitution $x = \frac{Hk_S^2\sqrt{d_3}}{4M}\tau^2$. We have also used that

$$\Gamma(3/4) = \frac{\sqrt{2}\pi}{\Gamma(1/4)}\ . \tag{93}$$

So, the contribution from the operator $\pi(\partial_i\partial_j\pi)^2$ for $k_1 = k_L \ll k_S = k_2 \simeq k_3$ is

$$\langle\pi_{k_1}\pi_{k_2}\pi_{k_3}\rangle_{\pi(\partial_i\partial_j\pi)^2} = -6\pi\delta^3\big(\sum_i k_i\big)\partial_t(d_3M^2)\frac{H\Gamma^4(3/4)}{2M^2(d_3M^2)^{5/2}k_S^3k_L^3}\ , \tag{94}$$

Next let us compute the contribution from the operator $\pi\dot{\pi}^2$ in the same limit. This is very similar, we just have a slightly different integral

$$\begin{aligned} \langle\pi_{k_1}\pi_{k_2}\pi_{k_3}\rangle_{\pi\dot{\pi}^2} &= i16\dot{M}M^3(2\pi)^3\delta^3\big(\sum_i k_i\big)\pi^{cl}_{k_1}(0)\pi^{cl}_{k_2}(0)\pi^{cl}_{k_3}(0) \\ &\quad\times\int\frac{d\tau}{(H\tau)^2}\pi^{cl*}_{k_1}(\tau)\partial_\tau\pi^{cl*}_{k_2}(\tau)\partial_\tau\pi^{cl*}_{k_3}(\tau) + c.c. \\ &\simeq -i16\dot{M}M^3(2\pi)^3\delta^3\left(\sum_i k_i\right)\frac{\Gamma^4(3/4)}{2HM^2(d_3M^2)^{3/2}\pi^2k_L^3k_S^3}\int d\tau\tau^{-2}\left(\partial_\tau\pi^{cl*}_{k_S}(\tau)\right)^2\ , \end{aligned} \tag{95}$$

where again we have used the long wavelength expansion for $\pi^{cl}_{k_1}$ in (87). The remaining integral gives:

$$\begin{aligned} I_{\pi\dot{\pi}^2} &= \int d\tau\,\tau^{-2}\left(\partial_\tau\pi^{cl*}_{k_S}(\tau)\right)^2 \\ &= \frac{\pi H^2}{16M^4}\left[\int dx\,x\left(H_{-1/4}(x)\right)^2\right]^* \\ &= \frac{H^2}{32M^4}(1-i)\ , \end{aligned} \tag{96}$$

so that the result for this term after adding the complex conjugate is

$$\langle\pi_{k_1}\pi_{k_2}\pi_{k_3}\rangle_{\pi\dot{\pi}^2} = -4\pi\delta^3\big(\sum_i k_i\big)\frac{\dot{M}H\Gamma^4(3/4)}{M^3(d_3M^2)^{3/2}k_L^3k_S^3}\ . \tag{97}$$

Thus the full result in the squeezed limit is $(k_1 = k_L \ll k_S = k_2 \simeq k_3)$

$$\langle \pi_{k_1}\pi_{k_2}\pi_{k_3}\rangle = -2\pi\delta^3\big(\sum_i k_i\big)\left(\frac{3\partial_t(d_3M^2)}{d_3M^2}+4\frac{\dot{M}}{M}\right)\frac{H\Gamma^4(3/4)}{2M^2(d_3M^2)^{3/2}k_1^3k_2^3}, \tag{98}$$

converting this using $\zeta_k = -H\pi_k$ gives

$$\langle \zeta_{k_1}\zeta_{k_2}\zeta_{k_3}\rangle = 2\pi\delta^3\big(\sum_i k_i\big)\left(4\frac{\dot{M}}{M}+3\frac{\partial_t(d_3M^2)}{d_3M^2}\right)\frac{H^4\Gamma^4(3/4)}{2M^2(d_3M^2)^{3/2}k_L^3k_S^3}, \tag{99}$$

where $k_L = k_1$ and $k_S = k_2 \simeq k_3$. The consistency relation is

$$\lim_{k_1\to 0}\langle \zeta_{k_1}\zeta_{k_2}\zeta_{k_3}\rangle = -(2\pi)^3\delta^3\big(\sum_i k_i\big)n_s P_{k_L}P_{k_S}; \tag{100}$$

so, in order to verify it, we need to compute the right hand side of this equation using eq.s (89) and (90):

$$\begin{aligned}
-(2\pi)^3\delta^3\big(\sum_i k_i\big)n_s P_{k_L}P_{k_S} &= (2\pi)^3\delta^3\big(\sum_i k_i\big)\left(\frac{\dot{M}}{HM}+\frac{3\partial_t(d_3M^2)}{4Hd_3M^2}\right)\frac{(2\pi)^2H^5}{2M^2(d_3M^2)^{3/2}\Gamma^4(1/4)k_S^3k_L^3}\\
&= 2\pi\delta^3\big(\sum_i k_i\big)\left(4\frac{\dot{M}}{M}+3\frac{\partial_t(d_3M^2)}{d_3M^2}\right)\frac{H^4\Gamma^4(3/4)}{2M^2(d_3M^2)^{3/2}k_S^3k_L^3}.
\end{aligned} \tag{101}$$

We see that the consistency relation is again satisfied.

# 4 Conclusions

Observation of the non-Gaussian component of the CMB is going to improve rapidly in the next few years with the launch of the Planck satellite. This will reduce the current limit from WMAP 3yr data [11] by a factor of around 6 [25]. While standard slow roll inflation predicts a level on non-Gaussianity far beyond current sensitivity, there are many models of inflation which predict a larger level on non-Gaussianity and that are already beginning to be constrained by current observations.

For this reason, we consider it very important to understand the properties of the non-Gaussian signal coming from inflation. Along this line of reasoning, in [1, 15] it was pointed out that in all models of inflation with only one relevant degree of freedom, which acts as the clock of the system, the 3-point function in a particular geometrical limit, the squeezed triangle limit, should follow a consistency relation:

$$\lim_{k_1\to 0}\langle \zeta_{\vec{k}_1}\zeta_{\vec{k}_2}\zeta_{\vec{k}_3}\rangle = -(2\pi)^3\delta^3\big(\sum_i \vec{k}_i\big)P_{k_1}P_{k_3}\frac{d\log k_3^3P_{k_3}}{d\log k_3}\ , \tag{102}$$

This consistency relation involves a level of non-Gaussianity too small to be detectable by foreseeable experiments. However, it is still very important for the following reason: if we detect some signal in the squeezed triangle configuration, it will mean that all the single clock inflationary model will be ruled out.

Since this is a very powerful statement, we think it is very important to be sure that the consistency relation (102) is true. The purpose of the present paper has been to further settle this

issue. In the first part of the paper, we have made more explicit and rigorous the proof that was already present in [1, 15]. Still this proof turns out to be rather implicit, even though physically clear. For this reason, we found it useful to prove the consistency relation in a completely orthogonal way, i.e. through a direct check of all possible single field models. Clearly, this task seems at first rather difficult and ill defined. However, we have been able to do this because we could exploit a recently developed effective field theory for inflation [16, 17] which completely describes the fluctuations around an inflationary background under the very general hypothesis that there is only one dynamical system and that this spontaneously breaks time diffeomorphisms.

The calculations are a bit convoluted, even though we have been able to use some simplifying tricks, and for some complicated models we have performed the study only in some simple limit. However, the result is very immediate: the consistency relation of the 3-point function of all single clock models *does* hold, and hopefully this will help us learn something new from the upcoming cosmological experiments.

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# Appendix

# A Matching our Unitary Gauge Lagrangian to a general $P(X,\phi)$ Lagrangian

As one should have expected, our effective Lagrangian reproduces the result of the widely used k-inflation Lagrangian [21], as we are now going to verify.

The Lagrangian of k-inflation can be written in the form:

$$S_{\mathrm{k-inf}} = \int d^4x \sqrt{-g} P(X,\phi) \ , \tag{103}$$

where $X = -g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. To match our effective Lagrangian to the k-inflation Lagrangian, we just write the k-inflation Lagrangian in unitary gauge $\phi(\vec{x},t) = \phi_0(t)$. The $X$ variable becomes:

$$X = \frac{\dot{\phi}_0(t)^2}{N^2} \tag{104}$$

Obviously, the factor $1/N^2$ can only enter the unitary gauge Lagrangian in terms with $X_0(t) = \dot{\phi}_0^2$. For this reason, an expansion of $S_{\mathrm{k-inf}}$ in powers of $1/N^2 - 1$ is the same as an expansion in powers of $X - \dot{\phi}_0^2$. We obtain:

$$c_n(t)\ M^4(t) \quad = \quad \dot{\phi}_0(t)^{2n} \frac{\partial^n}{\partial X^n} P(X,\phi)\big|_0 \ . \tag{105}$$

where in our Lagrangian $c_2(t) \equiv 1$, and $c_1(t) = -M_{\rm Pl}^2 \dot{H}$ and $c_0(t) = -M_{\rm Pl}^2(3H^2 + 2\dot{H})$.

In particular, we find that the speed of sound is

$$c_s^{-2} = 1 - \frac{2M(t)^4}{\dot{H}M_{\rm Pl}^2} \tag{106}$$

$$= 1 + 2\dot{\phi}_0^2 \frac{\partial^2 P}{\partial X^2}\Big|_0 \left(\frac{\partial P}{\partial X}\Big|_0\right)^{-1} \tag{107}$$

In general the use of a polynomial as $P(X, \phi)$ is not consistent in a regime of effective field theory, because the function $P$ involves many non-renormalizable terms. However, it is possible that some UV complete theory can induce a low energy effective theory with the structure of k-inflation, where a precise relationship between all the non-renormalizable operators is understood. This is the case for DBI inflation, a scenario in which the inflaton represents the coordinate of a brane traversing an extra dimensional warped geometry. Due to the causal speed limit in the extra dimension, the inflaton cannot roll arbitrarily fast, and in fact experiences a relativistic drag that admits slow roll behavior despite a steep potential [3]. This effect can also be understood in terms of the purely four dimensional CFT dual because as the inflaton rolls, new light scalar modes become accessible. Integrating out these modes induces a frictional force encapsulated by the low energy Lagrangian [3]

$$S_{\rm DBI} = \int d^4x\sqrt{-g}\left[-f^{-1}(\phi)\sqrt{1 - f(\phi)X} - V(\phi)\right]. \tag{108}$$

Notice the resemblance of this Lagrangian to that of a relativistic point particle in an external potential. Indeed, following this analogy we see that without the potential the DBI Lagrangian possesses a residual 5D spacetime symmetry corresponding to arbitrary "boosts" in the $\phi$ direction. Such a transformation rotates the $d\phi$ direction into the $dt$ time direction, and so it acts nonlinearly on derivatives of the inflaton

$$\dot{\phi} \rightarrow \dot{\phi} + \eta(f^{-1/2} - f^{1/2}\dot{\phi}^2), \tag{109}$$

$$dt \rightarrow dt(1 + \eta f^{1/2}\dot{\phi}), \tag{110}$$

where $\eta$ is an infinitesimal boost parameter. In the UV, this symmetry comes from the isometries of AdS$_4$, or, dually, from superconformal invariance of $\mathcal{N} = 4$ SYM [22].

From the the nonlinear form of this boost, it is clear that coefficients of higher order terms in $\dot{\phi}$ are uniquely set by those of lower order terms. Consequently, the square root form of the DBI action is uniquely set by this symmetry, and moreover we find that

$$c_n(t)M^4(t) \sim \frac{\dot{\phi}_c^{2n} f(\phi_c)^{n-1}}{(1 - f\dot{\phi}_c^2)^{n-1/2}}, \quad n \geq 2 , \tag{111}$$

where we have neglected some numerical coefficients. Since the FRW equations set the value of $c(t)$ and $\Lambda(t)$, and the speed of sound sets the ratio of $M(t)^4$, as usual we can rewrite $\dot{\phi}_0^2$ and $f(\phi_0)$ in terms of $H$, $c_s$, and slow roll parameters, giving

$$\dot{\phi}_0^2 = 2\epsilon H^2 M_{\rm Pl}^2 c_s, \tag{112}$$

$$f(\phi_0) = \frac{c_s}{2\epsilon H^2 M_{\rm Pl}^2}\left(\frac{1}{c_s^2} - 1\right). \tag{113}$$

where $\epsilon = -\dot{H}/H$. Plugging these expressions back into (111), we obtain

$$c_n(t)M(t)^4 \sim \epsilon H^2 M_{\rm Pl}^2 \left(\frac{1}{c_s^2} - 1\right)^{n-1} \quad , \quad n \geq 2 \ , \tag{114}$$

so we see that the symmetries of the DBI action demand a very particular form for the the coefficients of our effective Lagrangian. In particular, we can also see that, in the limit of $c_s \ll 1$, the symmetries impose an hierarchy between the $c_n$'s.

# B Non Linear Relation between $\pi$ and $\zeta$

We will work out the relationship between the $\pi$ gauge (24) and the $\zeta$ gauge (6) using the results of [1]. Since $\zeta$ is the relevant variable which is constant outside the horizon, we need to know $\zeta$ in terms of $\pi$ in order to determine the observational consequences of our effective Lagrangian. Let us remind ourselves which gauge properly defines the $\zeta$ variable. This is not the spatially flat slicing we used in deriving the next to leading order Lagrangian in $\pi$ eq. (24), but it is rather a gauge fixed version of the unitary gauge we used to build our effective Lagrangian. In $\zeta$ gauge time diffs are fixed by imposing precisely that $\pi = 0$, while spatial diffs are fixed by requiring the spatial metric to be isotropic. In other words, the $\zeta$ gauge is defined by the condition:

$$\pi = 0 \ , \qquad \hat{g}_{ij} = a^2 e^{2\zeta} \delta_{ij} \ . \tag{115}$$

If we denote by $\tilde{t}$ the time coordinate in the $\pi$ gauge, and by $t$ the time in the $\zeta$ gauge, we have to perform a time reparametrization of the form $\tilde{t} = t + T(x)$ to go from $\pi$ gauge to the $\zeta$ gauge. We have then to solve the following equation:

$$\pi(\tilde{x}) \to \pi_\zeta(x(\tilde{x})) = \pi(\tilde{x}) + T(x(\tilde{x})) = 0 \tag{116}$$

where in the second passage we have used the fact that the $\pi_\zeta$, *i.e.* the $\pi$ in $\zeta$ gauge, is zero. This implies that

$$T(x) = -\pi(x) + \dot{\pi}(x)\pi(x) \tag{117}$$

to second order. Maldacena [1] worked out $\zeta$ in terms of $T(x)$; to quadratic order he found

$$\begin{aligned} \zeta &= HT + \frac{1}{2}\dot{H}T^2 + \alpha(T) && (118) \\ &= -H\pi + H\dot{\pi}\pi + \frac{1}{2}\dot{H}\pi^2 + \alpha(T(\pi)) && (119) \end{aligned}$$

where $\alpha$ is determined by solving for the additional spatial diffeomorphism needed to maintain the $\zeta$ gauge condition. Although the $\alpha$ term is of second order in $T$, it only contains higher derivative terms that vanish outside the horizon, so it is irrelevant for our calculation.

# C Wave Equation at First Order in Slow Roll for the case $\omega^2 = c_s^2 k^2$

Using the conformal time $d\tau = dt/a$, the quadratic action $S_2[\pi]$ is

$$
\begin{aligned}
S_2 &= \frac{1}{2}\int d^3x d\tau (a^2 f^2)\left[\frac{\pi'^2}{a^2} - \frac{c_s^2}{a^2}(\partial\pi)^2 - m^2\pi^2\right] + \dots, && (120)\\
f^2 &= -\frac{2a^2 \dot{H} M_{\rm Pl}^2}{c_s^2}, && (121)\\
m^2 &= 3\dot{H}, && (122)
\end{aligned}
$$

where dots and primes denote derivatives with respect to $t$ and $\tau$, respectively, and $\tau$ runs from $-\infty$ to 0. Applying the field redefinition

$$
\pi = f^{-1}\sigma, \tag{123}
$$

and going to Fourier space, we obtain a wave equation for $\sigma_k(\tau)$,

$$
\begin{aligned}
\sigma_k'' + c_s^2 k^2 \sigma_k &= \left(-a^2 m^2 + \frac{f''}{f}\right)\sigma_k && (124)\\
&= 2a^2 H^2\left(1 - \frac{\epsilon}{2} + \frac{3\eta}{4} - \frac{3s}{2}\right)\sigma_k. && (125)
\end{aligned}
$$

Solving the wave equation, and imposing to be in the Minkowski vacuum at early times, we find that classically

$$
\begin{aligned}
\pi_k^{\rm cl}(\tau) &\sim (-\tau)^{3/2} H_\nu^{(1)}(-c_s k\tau(1+s)), && (126)\\
\nu &= \frac{3}{2} + \epsilon + \frac{\eta}{2} + \frac{s}{2}. && (127)
\end{aligned}
$$

We determine the overall normalization of this function when we quantize $\pi$.

We promote $\pi_k$ to a field operator

$$
\pi_k(t) = \pi_k^{cl}(t) a_k + \pi_{-k}^{cl*}(t) a_{-k}^\dagger, \tag{128}
$$

where $\pi_k^{cl}$ satisfies the wave equation and has normalization set by the canonical commutation relation

$$
[a_k, a_{k'}^\dagger] = (2\pi)^3 \delta^{(3)}(k - k'), \tag{129}
$$

yielding

$$
\begin{aligned}
\pi_k^{cl}(\tau) &= -\sqrt{\frac{\pi}{8\epsilon}}\frac{c_s}{M_{\rm Pl}}(-\tau)^{3/2}(1 - \epsilon_* + s_*/2) e^{i\pi(\epsilon_*/2 + \eta_*/4)} H_\nu^{(1)}(-c_s k\tau(1+s)), && (130)\\
\nu &= \frac{3}{2} + \epsilon + \frac{\eta}{2} + \frac{s}{2}. && (131)
\end{aligned}
$$

We follow [4], and define $\epsilon_*$, $\eta_*$ and $s_*$ as the slow roll parameters evaluated at the time at which $K_1 = k_1 + k_2 + k_3$ exits the horizon, so that $c_s K_1 = aH$. One can explicitly calculate the tilt

of the power spectrum for $\zeta$ by expanding $|\zeta_k|^2 = |H\pi_k(\tau)|^2$ in the late time limit, $k \to 0$. In this regime, the Hankel function behaves as $H_\nu(k) \propto k^{-\nu}$, so the two-point correlator behaves as $k^{-3-2\epsilon-\eta-s} \equiv k^{-3+n_s}$. This shows very explicitly that

$$n_s = -2\epsilon - \eta - s \ . \tag{132}$$

Setting to zero the slow roll parameters, we recover the de Sitter mode

$$\pi_k^{cl}(\tau) \quad = \quad \frac{i}{2M_{\rm Pl}k^{3/2}\sqrt{c_s\epsilon}}(1+ic_sk\tau)e^{-ic_sk\tau}. \tag{133}$$

## C.1 Explicit Long Wavelength Behavior of $\dot{\zeta}$

One of the major advantages of the $\zeta$ variable is that it goes exponentially fast to a constant when it is outside of the horizon. In conformal time, this exponential speed turns into a power law behavior. This is clearly manifest in the classical $\zeta$ modes at leading order in the slow roll parameters, where $\zeta = -i\frac{H}{2M_{\rm Pl}\sqrt{\epsilon c_s k^3}}(1+ic_sk\tau)e^{-ic_sk\tau}$, but not at higher orders. However, at next to leading order, in the long wavelength limit, the classical modes can be written in closed form.

To begin, we expand $H^{(1)}_\nu(x) = J_\nu(x) + iY_\nu(x)$ around $\nu = 3/2$ and $x = 0$:

$$H^{(1)}_\nu(x) = -\sqrt{\frac{2}{\pi}}i\frac{(1-ix)e^{ix}}{x^{3/2}}\left(1-(\nu-\frac{3}{2})\left[\gamma-2+\log 2+\log x\right]\right) \tag{134}$$

The constant $\gamma = 0.577\ldots$ is the Euler-Mascheroni constant. Inserting this in our expression for the classical $\pi$ modes in equation (131), we obtain, dropping the irrelevant constant phase:

$$\pi_k^{cl}(\tau) = \frac{i}{2}\frac{\epsilon^{-1/2}c_s^{-1/2}}{M_{\rm Pl}k^{3/2}}(1+\frac{1}{2}(c_sk\tau(1+s))^2)(1-\epsilon_*+\frac{s_*}{2}-\frac{3s}{2})(1-(\epsilon+\frac{\eta}{2}+\frac{s}{2})(\gamma-2+\log 2+\log(-c_sk\tau(1+s)))) \tag{135}$$

The additional time-dependence that appears at next-to-leading order in slow-roll is the $\log\tau$ component as well as the time-dependence of the parameters $c_s, \epsilon, \eta, s$, themselves. When we differentiate with respect to time, the derivatives of slow-roll parameters will be present at next to leading order only for parameters that were already present in the leading order modes. With this in mind, it is straightforward to calculate that

$$\dot{\zeta}_k = i\frac{H^2\tau^2}{2M_{\rm Pl}}\sqrt{\frac{c_s^3k}{\epsilon}}\left(1+(2\epsilon-\epsilon_*+\eta+\frac{s}{2}+\frac{s_*}{2})+(\epsilon+\frac{\eta}{2}+\frac{s}{2})(-\gamma-\log(-2k\tau c_s))\right) \tag{136}$$

Individual terms in $H\dot{\pi}$ or $\dot{H}\pi$ proportional to negative powers of $k$ have cancelled out.

# D Proof that $\zeta$ is Constant Outside the Horizon

In this Appendix we generalize Maldacena's proof [1] (see also [26]) that $\zeta$ is constant outside the horizon to a completely general model of inflation with one degree of freedom. The idea of the proof is simple – we will expand the action to first order in *derivatives* of the dynamical field $\zeta$, but to all orders in $\zeta$ itself, without any other approximations. We will show that to this order, the action is a total derivative, so the $\zeta$ action must begin at second order in derivatives. Thus $\dot{\zeta} = 0$ is

always a solution of the equations of motion when we neglect spatial derivatives. Moreover, we shall briefly show that the constant solution is an attractor, in the sense that solutions in its neighborhood approach it exponentially fast in time. As we show in the main text, the solution to the linearized $\zeta$ equation which deep inside the horizon is in the Minkowski vacuum, at late times asymptotes exponentially to the $\zeta = \text{const}$ solution [11]. Since we are always in the perturbative regime this tells us that we are in the basin of attraction of the constant solution, and that therefore $\zeta$ is constant outside of the horizon at non linear level.

The first step of the proof is to show self-consistently that $\delta N, \hat{\nabla}_i N_j = \mathcal{O}(\partial_\mu \zeta)$ [12]. This makes sense intuitively – since $N$ and $\hat{\nabla}_j N_i$ are constrained variables, while $\zeta$ is the only dynamical degree of freedom, if derivatives of $\zeta$ were to vanish, then we would simply have a pure FRW cosmology up to a trivial rescaling of the coordinates, and $\delta N$ and $\hat{\nabla}_i N_j$ would also vanish. Thus they must be proportional to derivatives of $\zeta$. If we assume that this is the case, then we only need the action to quadratic order in $\delta N$ and $\hat{\nabla}_i N_j$. Thus for our purposes, the relevant piece of the action is

$$S = \int d^3x dt \sqrt{\hat{g}} \left[ N \frac{M_{\rm Pl}^2}{2} R^{(3)} + \frac{1}{N} \frac{M_{\rm Pl}^2}{2} \left( E_{ij} E^{ij} - E_i^{i\,2} \right) - \frac{1}{N} M_{\rm Pl}^2 \dot{H} - N M_{\rm Pl}^2 (3H^2 + \dot{H}) \right.$$
$$\left. + \frac{1}{2} N M(t)^4 \left( \frac{1}{N^2} - 1 \right)^2 + \frac{d_1}{2} M(t)^3 N \delta N \delta E_i^i - \frac{d_2(t)}{2} M(t)^2 \delta E^i{}_i{}^2 - \frac{d_3(t)}{2} M(t)^2 \delta E^i{}_j \delta E^j{}_i \right]. \quad (141)$$

Now since $N$ is simply a Lagrange multiplier, we can vary the action with respect to it to obtain its (algebraic) equation of motion

$$2\delta N \left( 3M_{\rm Pl}^2 H^2 + M_{\rm Pl}^2 \dot{H} - 2M^4 \right) = \dot{\zeta} \left( 6M_{\rm Pl}^2 H + \frac{3}{2} d_1 M^3 \right) + \hat{\nabla}_i N^i \left( -2M_{\rm Pl}^2 H - \frac{d_1}{2} M^3 \right) \quad (142)$$

where we have used the fact that

$$E_{ij} E^{ij} - E_i^{i\,2} = -6 \left( H + \dot{\zeta} \right)^2 + 4 \left( H + \dot{\zeta} \right) \hat{\nabla}_i N^i + \mathcal{O}\left( (\hat{\nabla}_i N^i)^2 \right). \quad (143)$$

Analogously for $\hat{\nabla}_j N_i$ we obtain:

$$\frac{1}{2} M_{\rm Pl}^2 \left[ 4\hat{\nabla}_i \left( \dot{\zeta} - H\delta N \right) + \hat{\nabla}_j \hat{\nabla}^j N_i + \hat{\nabla}_j \hat{\nabla}_i N^j - 2\hat{\nabla}_i \hat{\nabla}_j N^j \right] - \frac{d_1}{2} M(t)^3 \hat{\nabla}_i \delta N + \quad (144)$$
$$d_2 M(t)^2 \left( 3\hat{\nabla}_i \dot{\zeta} - \hat{\nabla}_i \hat{\nabla}_j N^j \right) + \frac{d_3}{2} M(t)^2 \left( 2\hat{\nabla}_i \dot{\zeta} - \hat{\nabla}_j \hat{\nabla}^j N_i - \hat{\nabla}_j \hat{\nabla}_i N^j \right) = 0$$

[11] The fact that one of the two solutions to the linearized equation is exponentially damped outside of the horizon is very general: in the $\pi$ equation, it comes from the friction term $3H\dot{\pi}$ where $\pi = -\zeta/H$ at linear level.

[12] At linear order the solutions for $N$ and $N^i$ in the gauge of $\zeta$ are given by:

$$N - 1 = \frac{A_1}{C} \dot{\zeta} - \frac{B_1}{C} (\partial^2 \zeta / a^2) \quad , \quad \partial_i N^i = \left( \frac{A_2}{C} \dot{\zeta} + \frac{B_2}{C} (\partial^2 \zeta / a^2) \right) \ , \quad (137)$$

where, up to small terms suppressed by $M/M_{\rm Pl} \ll 1$ or $H/M \ll 1$, the coefficients are:

$$A_1 = 4M_{\rm Pl}^2 \left( d_1 M^3 - 4M_{\rm Pl}^2 H \right) \quad , \quad B_1 = 8M_{\rm Pl}^2 \left( d_2 + d_3 \right) M^2 \ , \quad (138)$$
$$A_2 = 16 M_{\rm Pl}^2 \left( -2M^4 + M_{\rm Pl}^2 \dot{H} \right) \quad , \quad B_2 = 4M_{\rm Pl}^2 \left( d_1 M^3 + 4M_{\rm Pl}^2 H \right) \ , \quad (139)$$
$$C = \left( d_1^2 - 16 \left( d_2 + d_3 \right) \right) M^6 - 16 M_{\rm Pl}^4 H^2 \ . \quad (140)$$

The linear solution for $\zeta$ vanishes outside of the horizon quickly enough so that $\delta N$ and $N^i$ go to zero in the same limit.

We see that it is self-consistent to assume that $\delta N, \hat{\nabla}_i N_j = \mathcal{O}(\partial_\mu \zeta)$, which justifies our neglect of higher powers of these parameters.

Now let us expand the action to linear order in $\partial_\mu \zeta$. For this purpose, note that $R^{(3)}$ is of quadratic order, so we can neglect it. We find

$$S = \int d^3x dt\, a(t)^3 e^{3\zeta} M_{\rm Pl}^2 \left[\frac{1}{2}(1-\delta N)\left(-6\left(H^2 + 2H\dot{\zeta}\right) + 4H\hat{\nabla}_i N^i\right) - (1-\delta N)\dot{H} \right.$$
$$\left. -(1+\delta N)\left(3H^2 + \dot{H}\right)\right], \qquad (145)$$

if we simplify and integrate the $\hat{\nabla}_i N^i$ term by parts, we find

$$\begin{aligned} S &= \int d^3x dt\, a(t)^3 e^{3\zeta} M_{\rm Pl}^2 \left[-6H^2 - 2\dot{H} - 6H\dot{\zeta}\right] \\ &= \int d^3x dt \frac{d}{dt}\left[-a(t)^3 e^{3\zeta} 2H M_{\rm Pl}^2\right]. \end{aligned} \qquad (146)$$

Note that the term linear in $\delta N$ dropped out of the action. This is not surprising – it is due to the fact that $\frac{\partial L}{\partial(\delta N)} = 0$ is satisfied to zeroth order in derivatives, simply because the metric satisfies Einstein's equations, and we have neglected terms at second order in derivatives.

As claimed, the action for $\zeta$ begins at second order in derivatives for all models of inflation based on a single degree of freedom. This means that $\zeta =$ const is always a solution of the equations of motion when we neglect gradients. However, the solution we are interested in, which asymptotes to the Minkowski vacuum deep inside the horizon, is not exactly constant. At linear level, the time dependent component goes to zero exponentially fast in time. We want to verify this at non linear level, by showing that in fact the $\zeta =$ const solution is still an attractor.

This can be done rather quickly by the following reasoning. Let us consider a small perturbation around a solution $\zeta = \zeta_0 =$ const:

$$\zeta = \zeta_0 + \psi\ , \qquad (147)$$

and let us ask what is the linearized equation of motion for $\psi$. By the same definition of $\zeta$ (eq.(115)), a constant component $\zeta_0$ can be reabsorbed in the definition of $a(t)$: $\tilde{a}(t) = a(t)\ e^{\zeta_0}$. This implies that the linearized equation of motion of $\psi$ in a $\zeta_0$ background has to be the same as the one of $\zeta$, with $a(t)$ replaced with $\tilde{a}(t)$. Therefore the solution for $\psi$ is the same as for the linearized $\zeta$: out of the horizon it either decays exponentially fast in cosmic time, or it goes to a constant (and in this case it amounts at just a redefinition of $\zeta_0$).

We therefore conclude that outside of the horizon the solution for $\zeta$ quickly converges to a constant and $N$ and $N^i$ tend to their unperturbed values. At this point, via the argument of Sec. 2, we deduce that the consistency relation between the three point function of $\zeta$ and the tilt of the spectrum must hold for the models with only one relevant degree of freedom.

# E Reintroducing the $\pi$ in $\delta E^i{}_i$

Since the reintroduction of the $\pi$ field in $\delta E^i{}_i$ is not entirely trivial, in this appendix we perform an explicit calculation. In the spatially flat gauge (24) in which we are working, the trace of $\delta E_{ij}$ can be written as:

$$\delta E^i_i = \tfrac{1}{2}\hat{g}^{ij}\partial_t \hat{g}_{ij} - \partial_i N^i - 3H \qquad (148)$$

The reintroduction of the $\pi$ field follows the same steps we illustrate of [17]. What makes this case slightly more complicated than the case explicitly illustrated in [17] is the fact that the transformation of $\hat{g}_{ij}$ and $N_i$ under time diffeomorphisms are more complicated than the one of $N$. The following equations are true:

$$
\begin{aligned}
g^{0i} &= \frac{N^i}{N^2} \\
-g^{00} &= \frac{1}{N^2}\,.
\end{aligned}
\tag{149}
$$

where $g_{\mu\nu}$ is the full 4d metric. Taking variations of $g^{0i}$, we find

$$
\delta g^{0i} = \frac{1}{N^2}\delta N^i + N^i\delta(\frac{1}{N^2}) + \delta N^i\delta(\frac{1}{N^2})\,. \tag{150}
$$

Solving for $\delta N^i$, and using $\frac{1}{N^2} = -g^{00}$, $N^i = -g^{0i}/g^{00}$, we eventually arrive at a formula for the variation of $N^i$ in terms of only the metric and its variations:

$$
\delta N^i = -(\delta g^{0i} - \frac{g^{0i}}{g^{00}}\delta g^{00})(g^{00}+\delta g^{00})^{-1}\,. \tag{151}
$$

$g^{0i}$ has the transformation law

$$
\begin{aligned}
g^{0i} &\to g^{\alpha\beta}\left(\frac{\partial(t+\pi)}{\partial x^\alpha}\right)\left(\frac{\partial x^i}{\partial x^\beta}\right) \\
&= g^{0i} + g^{0i}\dot{\pi} + g^{ij}\partial_j\pi \\
\delta g^{0i} &= \frac{N^i}{N^2}\dot{\pi} + \hat{g}^{ij}\partial_j\pi\,.
\end{aligned}
\tag{152}
$$

We therefore find, to second order in the fields (which is all we need to get the cubic action)

$$
\begin{aligned}
\delta N^i &= -(\frac{N^i}{N^2}\dot{\pi} + \frac{\partial^i\pi}{a^2} - N^i\delta g^{00})(g^{00}+\delta g^{00})^{-1} \\
&= \partial^i\pi + 2\delta N\partial^i\pi + 3N^i\dot{\pi} - 2\partial^i\pi\dot{\pi}\,.
\end{aligned}
\tag{153}
$$

The variation of $\hat{g}^{ij}\partial_t\hat{g}_{ij}$ also takes some work. It is clear that the "spatial" components of the 4d metric $g^{ij}$ do not change under time diffs, but $g^{ij} = \hat{g}^{ij} - \frac{N^iN^j}{N^2}$, so $\hat{g}^{ij}$ transforms. Using our transformation laws for $N^i$ and $1/N^2$, we find to second order in the fields that

$$
\frac{N^iN^j}{N^2} \to \frac{N^iN^j}{N^2} + 2\partial^{(i}\pi N^{j)} + \partial^i\pi\partial^j\pi\,, \tag{154}
$$

where round brackets stand for symmetrization. This implies

$$
\hat{g}^{ij} \to \hat{g}^{ij} + 2\partial^{(i}\pi N^{j)} + \partial^i\pi\partial^j\pi\,, \tag{155}
$$

Incidentally, for terms at second order in the fields, we can raise and lower spatial indices with the unvaried $\hat{g}_{ij}$ and $\hat{g}^{ij}$, since corrections to this are cubic order in the fields. Also, we could have derived the above transformation of $\hat{g}_{ij}$ by calculating the transformation of the "spatial" components of the 4d metric $g_{ij}$. It is straightforward to see that both methods agree, since

$$
g_{ij} \to \frac{\partial x^\alpha}{\partial x'^i}\frac{\partial x^\beta}{\partial x'^j}g_{\alpha\beta}\,, \tag{156}
$$

and $\frac{\partial x^\alpha}{\partial x'^i}$ is the $\alpha\, i$ component of the inverse of the matrix

$$\frac{\partial x'^\alpha}{\partial x^\beta} \;=\; \left(\begin{array}{cc} 1+\dot\pi & \partial_i\pi \\ 0 & \mathbf{1} \end{array}\right)_{\alpha\beta} . \tag{157}$$

Here Greek indexes run from 0 to 3 while Latin indexes run from 1 to 3. Thus, to second order in the fields,

$$\hat g^{ij}\partial_t \hat g_{ij} \;\rightarrow\; \frac{6H}{1+\dot\pi} - \partial_t(2\partial_i\pi N^i + (\partial\pi)^2)\,, \tag{158}$$

and finally, we have

$$\begin{aligned} \delta E^i_i \;&\rightarrow\; \delta E^i_i - 3H\dot\pi + 3H\dot\pi^2 - \tfrac{1}{2}\partial_t(2\partial_i\pi N^i + (\partial\pi)^2) \\ &-\; \partial^2\pi - \partial_i(2\delta N\partial^i\pi + 3N^i\dot\pi - 2\partial^i\pi\dot\pi)\,. \end{aligned} \tag{159}$$

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